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A
TREATISE
OF
FLUXIONS.

In Two BOOKS.

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FLUXIONS.

In Two BOOKS.

BY

COLIN MACLAURIN, A. M.

Professor of Mathematics in the University of Edinburgh, *and Fellow of the Royal Society.*

VOLUME I.

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To His GRACE

The Duke of ARGYLE

AND

GREENWICH.

MY LORD,

IT is not the ſhining Figure your GRACE has made in the higheſt Employments, neither is it the Luſtre which your quitting them has added to your Character; it is not the Favour of Princes which you have often enjoy'd, nor the loud and univerſal Applauſes of the People which you at preſent poſſeſs; but it

is the ſteady Virtue which has conducted you throughout, that determines me to offer this Work to your GRACE. And, having been honoured with the Countenance and Favour of ſo great a Man, I embrace the Opportunity of expreſſing my Gratitude at a Time when I can have no other Motive than that I am, with Truth, and the utmoſt Reſpect,

MY LORD,

Your Grace's

much obliged,

moſt faithful,

and moſt humble Servant,

COLIN MAC LAURIN.

PREFACE.

A Letter published in the Year 1734, *under the Title of* the Analyst, *first gave Occasion to the ensuing Treatise, and several Reasons concurred to induce me to write on this Subject at so great length. The Author of that Piece had represented the Method of Fluxions as founded on false Reasoning, and full of Mysteries. His Objections seemed to have been occasioned in a great measure by the concise Manner in which the Elements of this Method have been usually described; and their having been so much misunderstood by a Person of his Abilities appeared to me a sufficient Proof that a fuller Account of the Grounds of them was requisite.*

Though there can be no Comparison made betwixt the Extent or Usefulness of the ancient and modern Discoveries in Geometry, yet it seems to be generally allowed that the Ancients took greater Care, and were more successful in preserving the Character of its Evidence entire. This determined me, immediately after that Piece came to my Hands, (and before I knew any thing of what was intended by others in answer to it,) to attempt to deduce those Elements

ments after the Manner of the Antients, from a few unexceptionable Principles, by Demonstrations of the strictest Form. In my first Essay of this Kind, I contented myself with demonstrating the principal Cases of the Propositions of the four first Chapters of the first Book, and of the first Chapter of the second Book of the following Treatise, nearly in the same Form in which they now appear. But when it was communicated to some Gentlemen, they expressed a Desire that the same Method of Demonstration should be extended to other Branches of this Theory, and that I should enlarge the Plan. While I proceeded in this Work, I perceived that some Rules were defective or inaccurate; that the Resolution of several Problems which had been deduced in a mysterious manner, by second and third Fluxions, could be compleated with greater Evidence, and less Danger of Error, by first Fluxions only; and that other Problems had been resolved by Approximations, when an accurate Solution could be obtained with the same or greater Facility. These with other Observations concerning this Method, and its Application, led me on gradually to compose a Treatise of a much greater Extent than I intended, or would have engaged in, if I had been aware of it when I began this Work, because my Attendance in the University could allow me to bestow but a small part of my Time in carrying it on. And as this has been the Occasion of my Delay in publishing it, so I hope it will serve for an Apology, if some Mistakes have escaped me in treating of such a Variety of Subjects, in a manner different from that in which they have been usually explained.

In

In the mean Time the Defence of the Method of Fluxions, and of the great Inventor, was not neglected. Besides an Answer to the Analyst *that appeared very early under the Name of* Philalethes Cantabrigiensis, (*for the Author had concealed his real Name, as the* Analyst *whom he opposed had done,*) *a second by the same Hand in Defence of the first, a Discourse by Mr.* Robins, *a Treatise of Sir* Isaac Newton's *with a Commentary by Mr.* Colson, *and several other Pieces were published on this Subject. After I saw that so much had been written upon it to so good Purpose, I was the rather induced to delay the Publication of this Treatise, till I could finish my Design. I accommodated my Definition of the Variation of Curvature in Chap.* xi. *to Sir* Isaac Newton's, *to prevent Mistakes, as I have observed in Article* 386. *but made no material Alteration in any thing else. The greatest Part of the first Book was printed in* 1737: *But it could not have been so useful to the Reader without the second; and I would recommend to him, (if he is not already acquainted with this Method,) to peruse the two first Chapters of the second Book, before the five last of the first; there being a few Passages in these which I could not well avoid, that will be better understood by one who has some Knowledge of the principal Rules of the Method of Computation delivered in the second Book.*

In explaining the Notion of a Fluxion, I have followed Sir Isaac Newton *in the first Book, imagining that there can be no Difficulty in conceiving Velocity wherever there is Motion; nor do I think that I have* de-

departed from his Sense in the second Book; and in both I have endeavoured to avoid several Expressions, which, though convenient, might be liable to Exceptions, and, perhaps, occasion Disputes. I have always represented Fluxions of all Orders by finite Quantities, the Supposition of an infinitely little Magnitude being too bold a Postulatum *for such a Science as Geometry. But, because the Method of Infinitesimals is much in use, and is valued for its Conciseness, I thought it was requisite to account explicitely for the Truth, and perfect Accuracy, of the Conclusions that are derived from it; the rather, that it does not seem to be a very proper Reason that is assigned by Authors, when they determine what is called the* Difference *(but more accurately the Fluxion) of a Quantity, and tell us, That they reject certain Parts of the Element, because they become infinitely less than the other Parts; not only, because a Proof of this Nature may leave some Doubt as to the Accuracy of the Conclusion; but because it may be demonstrated that those Parts ought to be neglected by them at any Rate, or that it would be an Error to retain them. If an Accountant, that pretends to a scrupulous Exactness, should tell us that he had neglected certain Articles, because he found them to be of small Importance; and it should appear that they ought not to have been taken into Consideration by him on that Occasion, but belong to a different Account, we should approve his Conclusions as accurate, but not his Reason. This Method, however, may be considered as an easy and ready Way of distinguishing what Parts of an Element are to be rejected, and which are to be retained, in determining the precise Fluxion of a* Quan-

Quantity, or the Rate according to which it increases or decreases.

After I found that this Treatise could not be conveniently contained in one Volume, I was obliged to reprint two Leaves (Pages 411. &c.) *that it might be divided into two. I have reprinted likewise the first Sheet, chiefly on account of several Errors of the Press that had got into it, and one other Leaf* (*p.* 244.) *for the sake of a Passage, the Omission of which possibly would have been misintepreted. There are some Demonstrations in the first Chapter of the first Book that might have been abridg'd, and some, perhaps, will appear unnecessary. I have mentioned the Reasons that induced me to insist so fully on those elementary Parts in Articles* 69, 104. 494. *and* 697.

Several Treatises have appeared while this was in the Press, wherein some of the same Problems have been considered, though generally in a different manner. I have had Occasion to mention most of them, in the last Chapter of the second Book; but had not there an Opportunity to take notice, that the Problem in 480. *has been considered by Mr.* Euler *in his Mechanics. In most of the Instances wherein my Conclusions did not agree with those given by other Authors, I have not mentioned their Names.*

If upon the whole, the Evidence of this Method be represented to the Satisfaction of the Reader, some of the abstruse Parts illustrated, or any Improvements of this useful Art be proposed, I shall be under no great Concern, though Exceptions may

be made to ſome Modes of Expreſſion, or to ſuch Paſſages of this Treatiſe as are not eſſential to the principal Deſign.

INTRO-

INTRODUCTION.

GEOMETRY is valued for its extenſive uſefulneſs, but has been moſt admired for its evidence; mathematical demonſtration being ſuch as has been always ſuppoſed to put an end to diſpute, leaving no place for doubt or cavil. It acquired this character by the great care of the old writers, who admitted no principles but a few ſelf-evident truths, and no demonſtrations but ſuch as were accurately deduced from them. The ſcience being now vaſtly enlarged, and applied with ſucceſs to philoſophy and the arts, it is of greater importance than ever that its evidence be preſerved perfect. But it has been objected on ſeveral occaſions, that the modern improvements have been eſtabliſhed for the moſt part upon new and exceptionable maxims, of too abſtruſe a nature to deſerve a place amongſt the plain principles of the ancient geometry: And ſome have proceeded ſo far as to impute falſe reaſoning to thoſe authors who have contributed moſt to the late diſcoveries, and have at the ſame time been moſt cautious in their manner of deſcribing them.

In the method of indiviſibles, lines were conceived to be made up of points, ſurfaces of lines, and ſolids of ſurfaces; and ſuch ſuppoſitions have been employed by ſeveral ingenious men for proving the old theorems, and diſcovering new ones, in a brief and eaſy manner. But as this doctrine was inconſiſtent with the ſtrict principles of geometry, ſo it ſoon appeared that there was ſome danger of its leading them into falſe concluſions: Therefore others, in the place of indiviſible, ſubſtituted infinitely ſmall diviſible elements, of which they ſuppoſed all magnitudes to be formed; and thus endeavoured to retain, and improve, the advantages that were derived from the former method for the advancement of geometry. After theſe came to be reliſhed, an infinite ſcale of infinites and infiniteſimals, (aſcending and deſcending always by infinite ſteps,) was imagined, and propoſed to be received into geometry, as of the greateſt uſe for penetrating into its abſtruſe parts. Some have argued for quan-

tities more than infinite; and others for a kind of quantities that are ſaid to be neither finite nor infinite, but of an intermediate and indeterminate nature.

This way of conſidering what is called the ſublime part of geometry has ſo far prevailed, that it is generally known by no leſs a title than the *Science*, *Arithmetic*, or *Geometry of infinites*. Theſe terms imply ſomething lofty, but myſterious; the contemplation of which may be ſuſpected to amaze and perplex, rather than ſatisfy or enlighten the underſtanding, in the proſecution of this ſcience; and while it ſeems greatly to elevate geometry, may poſſibly leſſen its true and real excellency, which chiefly conſiſts in its perſpicuity and perfect evidence: For we may be apt to reſt in an obſcure and imperfect knowledge of ſo abſtruſe a doctrine, as better ſuited to its nature, inſtead of ſeeking for that clear and full view we ought to have of geometrical truth; and to this we may aſcribe the inclination which has appeared of late for introducing myſteries into a ſcience wherein there ought to be none.

There were ſome, however, who diſliked the making much uſe of infinites and infiniteſimals in geometry. Of this number was Sir Isaac Newton (whoſe caution was almoſt as diſtinguiſhing a part of his character as his invention) eſpecially after he ſaw that this liberty was growing to ſo great a height. In demonſtrating the grounds of the method of fluxions * he avoided them, eſtabliſhing it in a way more agreeable to the ſtrictneſs of geometry. He conſidered magnitudes as generated by a flux or motion, and ſhowed how the velocities of the generating motions were to be compared together. There was nothing in this doctrine but what ſeemed to be natural and agreeable to the ancient geometry. But what he has given us on this ſubject being very ſhort, his conciſeneſs may be ſuppoſed to have given ſome occaſion to the objections which have been raiſed againſt his method.

When the certainty of any part of geometry is brought into queſtion, the moſt effectual way to ſet the truth in a full light, and

* De quadrat. curvarum.

and to prevent difputes, is to deduce it from axioms or firft principles of unexceptionable evidence, by demonftrations of the ftricteft kind, after the manner of the antient geometricians. This is our defign in the following treatife; wherein we do not propofe to alter Sir ISAAC NEWTON's notion of a fluxion, but to explain and demonftrate his method, by deducing it at length from a few felf-evident truths, in that ftrict manner: and, in treating of it, to abftract from all principles and poftulates that may require the imagining any other quantities but fuch as may be eafily conceived to have a real exiftence. We fhall not confider any part of fpace or time as indivifible, or infinitely little; but we fhall confider a point as a term or limit of a line, and a moment as a term or limit of time: Nor fhall we refolve curve lines, or curvilineal fpaces, into rectilineal elements of any kind. In delivering the principles of this method, we apprehend it is better to avoid fuch fuppofitions: but after thefe are demonftrated, fhort and concife ways of fpeaking, though lefs accurate, may be permitted, when there is no hazard of our introducing any uncertainty or obfcurity into the fcience from the ufe of them, or of involving it in difputes. The method of demonftration, which was invented by the author of fluxions, is accurate and elegant; but we propofe to begin with one that is fomewhat different; which, being lefs removed from that of the ancients, may make the tranfition to his method more eafy to beginners, (for whom chiefly this treatife is intended,) and may obviate fome objections that have been made to it.

But, before we proceed, it may be of ufe to confider the fteps by which the ancients were able, in feveral inftances, from the menfuration of right-lin'd figures, to judge of fuch as were bounded by curve lines: for as they did not allow themfelves to refolve curvilineal figures into rectilineal elements, it is worth while to examine by what art they could make a tranfition from the one to the other: And as they were at great pains to finifh their demonftrations in the moft perfect manner, fo by following their example, as much as poffible in demonftrating a method fo much more general than theirs, we may beft guard againft exceptions and cavils, and vary lefs from the old foundations of geometry.

They found, that ſimilar triangles are to each other in the duplicate ratio of their homologous ſides; and, by reſolving ſimilar polygons into ſimilar triangles, the ſame propoſition was extended to theſe polygons alſo. But when they came to compare curvilineal figures, that cannot be reſolved into rectilineal parts, this method failed. Circles are the only curvilineal plane figures conſidered in the elements of geometry. If they could have allowed themſelves to have conſidered theſe as ſimilar polygons of an infinite number of ſides, (as ſome have done who pretend to abridge their demonſtrations,) after proving that any ſimilar polygons inſcribed in circles are in the duplicate ratio of the diameters, they would have immediately extended this to the circles themſelves; and would have conſidered the ſecond propoſition of the twelfth book of the Elements as an eaſy corollary from the firſt. But there is ground to think that they would not have admitted a demonſtration of this kind. It was a fundamental principle with them, that the difference of any two unequal quantities, by which the greater exceeds the leſſer, may be added to itſelf till it ſhall exceed any propoſed finite quantity of the ſame kind: and that they founded their propoſitions concerning curvilineal figures upon this principle in a particular manner, is evident from the demonſtrations, and from the expreſs declaration of Archimedes, who acknowledges it to be the foundation upon which he eſtabliſhed his own diſcoveries *, and cites it as aſſumed by the ancients in demonſtrating all their propoſitions of this kind. But this principle ſeems to be inconſiſtent with the admitting of an infinitely little quantity or difference, which, added to itſelf any number of times, is never ſuppoſed to become equal to any finite quantity whatſoever.

They proceeded therefore in another manner, leſs direct indeed, but perfectly evident. They found that the inſcribed ſimilar polygons, by increaſing the number of their ſides continually approached to the areas of the circles; ſo that the decreaſing differences betwixt each circle and its inſcribed polygon, by ſtill further and further diviſions of the circular arches which the

* Δείκνυται γὰρ ὅτι πᾶν τμῆμα περιεχόμενον ὑπὸ εὐθείας καὶ ὀρθογωνίου κώνου τομᾶς ἐπίτριτον ἐστὶ τοῦ τριγώνου, &c. *Archimed. de quadr. parab. ad Dosith.*

the ſides of the polygons ſubtend, could become leſs than any quantity that can be aſſigned: and that all this while the ſimilar polygons obſerved the ſame conſtant invariable proportion to each other, *viz.* that of the ſquares of the diameters of the circles. Upon this they founded a demonſtration, that the proportion of the circles themſelves could be no other than that ſame invariable ratio of the ſimilar inſcribed polygons: of which we ſhall give a brief abſtract, that it may appear in what manner they were able, in this inſtance, and ſome others of the ſame nature, to form a demonſtration of the proportions of curvilineal figures, from what they had already diſcovered of rectilineal ones. And that the general reaſoning by which they demonſtrated all their theorems of this kind may more eaſily appear, we ſhall repreſent the circles and polygons by right lines, in the ſame manner as all magnitudes are expreſſed in the fifth book of the Elements.

Suppoſe the right lines AB and AD to repreſent the two areas of the circles that are compared together; and let AP, AQ repreſent any two ſimilar polygons inſcribed in theſe circles. By further continual ſubdiviſions of the circular arches which the ſides of the polygons ſubtend, the areas of the polygons increaſe, and may approach to the circles AB and AD ſo as to differ from them by leſs than any aſſignable meaſure; the triangle which is ſubducted from each ſegment at every new ſubdiviſion being always greater than the half of the ſegment. The polygons inſcribed in the two circles, as they increaſe, are ever in the ſame conſtant proportion to each other: and this invariable ratio of theſe polygons muſt alſo be the ratio of the circles themſelves. For, if it is not, let the ratio of the polygons AP and AQ to each other be, in the firſt place, the ſame as the ratio of the circle AB to any magnitude AE leſs than the circle AD; ſuppoſe the ſubdiviſions of the arches of the circle AD to be continued till the difference betwixt the circle and inſcribed polygon become leſs than ED, ſo that the polygon may be repreſented by Aq, greater than AE; and let Ap repreſent a polygon inſcribed in the circle AB, ſimilar to the po-

polygon Aq. Then, ſince AP is to AQ as AB is to AE by the ſuppoſition, and the polygon Ap is to the ſimilar polygon Aq as AP is to AQ; it follows, that AB is to AE as Ap is to Aq; and that the circle AB being greater than Ap, a polygon inſcribed in it, AE muſt be greater than Aq. But Aq is ſuppoſed to be greater than AE; and theſe being repugnant, it follows, that the polygon AP is not to the polygon AQ as the circle AB is to any magnitude (as AE) leſs than the circle AD. For the ſame reaſon AQ is not to AP as AD is to any magnitude (as AF) leſs than AB. From which it follows that we cannot ſuppoſe AP to be to AQ as AB is to any magnitude Ae greater than AD; becauſe

if we take AF to AB as AD is to Ae, AF will be leſs than AB, and AP will be to AQ as AF leſs than AB to AD; againſt what has been demonſtrated. It follows therefore that AP is not to AQ as AB is to any magnitude greater or leſs than AD; but that the ratio of the circles AB and AD to each other, muſt be the ſame as the invariable ratio of the ſimilar polygons AP and AQ inſcribed in them, which is the duplicate of the ratio of their diameters.

In the ſame manner the ancients have demonſtrated, that pyramids of the ſame height are to each other as their baſes, that ſpheres are as the cubes of their diameters, and that a cone is the third part of a cylinder on the ſame baſe and of the ſame height. In general, it appears from this demonſtration, that when two variable quantities, AP and AQ, which always are in an invariable ratio to each other, approach at the ſame time to two determined quantities, AB and AD, ſo that they may differ leſs from them than by any aſſignable meaſure, the ratio of theſe limits AB and AD muſt be the ſame as the invariable ratio of the quantities AP and AQ: and this may be conſidered as the moſt ſimple and fundamental propoſition in this doctrine, by which we are enabled to compare curvilineal ſpaces in ſome of the more ſimple caſes.

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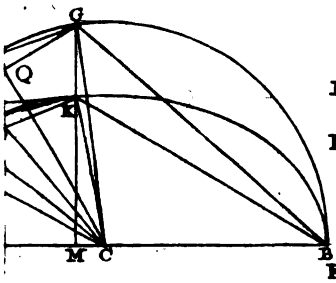
This general principle may ſerve for demonſtrating many other propoſitions, beſides the elementary theorems already mentioned. For example, let ADB be a ſemicircle deſcribed on the diameter AB, AEB a ſemiellipſe deſcribed on the ſame right line as its tranſverſe axis; let AFGB be any polygon deſcribed in the ſemicircle; and let FN, GM, perpendicular to AB, meet the ſemiellipſe in H and K, and the axis in N and M. Becauſe any ordinate of the circle is to the ordinate of the ellipſe on the ſame point of the axis as the tranſverſe axis is to the conjugate, it follows, that the triangle ANF is to the triangle ANH, the trapezium FNMG to the trapezium HNMK, the triangle GMB to the triangle KMB, and the whole polygon AFGB to the polygon AHKB, in the ſame conſtant ratio of the tranſverſe to the conjugate axis. Biſect the arch FG in D, and the triangle FDG will be greater than half the ſegment in which it is inſcribed. Let the ordinate DI meet the ellipſe in E, and the triangle HEK will be alſo greater than half the elliptic ſegment in which it is inſcribed; for it is obvious that the right lines GF, KH produced meet the axis in the ſame point R, and that the tangents at D and E meet it in the ſame point T; conſequently the tangent DT being parallel to the chord FG, the tangent of the ellipſe at E is parallel to HK, and the triangle HEK is greater than half the ſegment HEK. The polygons therefore AFGB, AHKB, by continually biſecting the circular arches, may approach to the areas of the ſemicircle and ſemiellipſe, ſo as to differ from them by leſs than any aſſignable meaſure. Hence if the right line AD in the preceeding article repreſent the area of the ſemicircle, AB the area of the ſemiellipſe, AQ the polygon inſcribed in the ſemicircle, AP the correſponding polygon inſcribed in the ſemiellipſe; it will appear, in the ſame manner, that the ſemicircle muſt be to the ſemiellipſe in the ſame ratio that is the conſtant proportion of thoſe inſcribed polygons, *viz.* that of the tranſverſe axis of the ellipſe to the conjugate axis. Fig. 1.

We have given this demonſtration a little different from that of Archimedes in his fifth propoſition of *conoids* and *ſpheroids*, that the ſame proportion might appear to be the ratio of the area

rea NFDGM in the circle to the area NHEKM in the ellipse. From which it follows, that if C be the common center of the circle and ellipse, the triangles CFN, CHN and the triangles CGM, CKM being to each other as the transverse axis is to the conjugate, the sector CFDG in the circle must be to the sector CHEK in the ellipse in the same proportion. Let the diameter CE meet its ordinate HK in P, and CP being to CE as CR is to CT, or as CQ is to CD; it appears, that when the ratio of CP to the semidiameter of the ellipse CE is given, then CQ is given, and therefore the sector CFDG is of a determined magnitude; consequently the elliptic sector CHEK, and the segment HEK, must each be of a determined invariable magnitude in the same ellipse, when the abscissis CP is in a given ratio to the semidiameter CE. The triangle CHK, and the trapezium CH*t*K (formed by the semidiameters CH, CK and the tangents H*t*, K*t*,) are also given in magnitude when this ratio of CP to CE, or of CE to C*t*, is given.

In general, if upon any diameter produced without the ellipse, any number of points be taken, on the same or on different sides of the center, at distances from it that are each in some given ratio to that diameter; and from these points tangents be drawn to the ellipse in any one certain order; the polygon formed by these tangents is always of a given magnitude in a given ellipse, and is equal to a polygon described by a similar construction about a circle, the diameter of which is a mean proportional betwixt the transverse and conjugate axis of the ellipse. The polygon inscribed in the ellipse by joining the points of contact, and the sectors bounded by the semidiameters drawn to these points, are also of given or determined magnitudes; and the Parts of any tangent intercepted betwixt the intersections of the other tangents with it, or betwixt these intersections and the point of contact, are always in the same ratio to each other in the same figure. There is an analogous property of the other conic sections.

When ARCHIMEDES demonstrated, that the area of a circle is equal to a triangle upon a base equal to the circumference of the circle, of a height equal to the radius, it was not by supposing



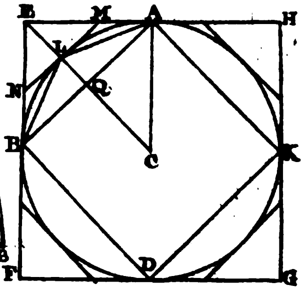


Fig. 2.

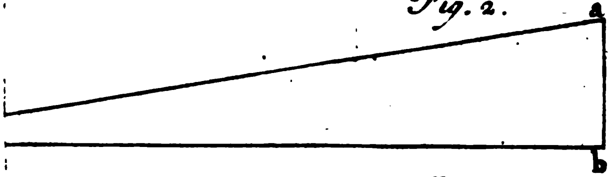


Fig. 4.

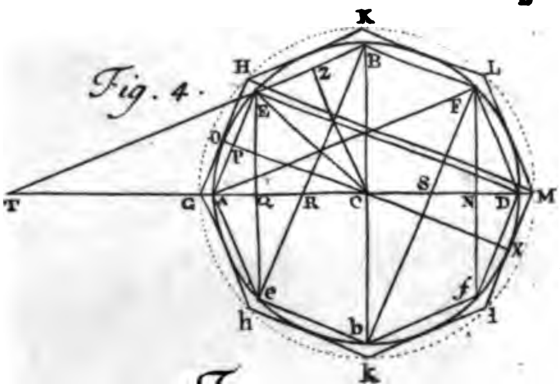
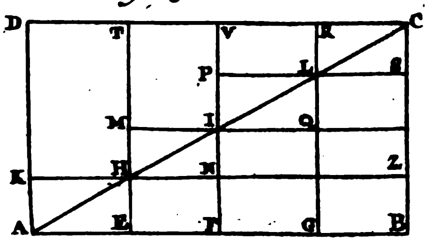


Fig. 5.

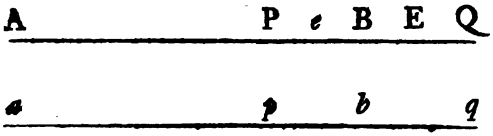


poſing it to coincide with a circumſcribed equilateral polygon of an infinite number of ſides, but in a more accurate and unexceptionable manner. Let *bd*, the baſe of the right-angled triangle *abd*, be ſuppoſed equal to the circumference of the circle ABD, *ab* equal to the radius CA, EFGH any equilateral polygon deſcribed about the circle, ABDK a ſimilar polygon inſcribed in it, and let CQ perpendicular to AB meet it in Q. As the circumſcribed polygon EFGH is greater than the circle, ſo it is greater than the triangle *abd*; becauſe it is equal to a triangle of a height equal to CA or *ab*, upon a baſe equal to the perimeter EFGH, which is always greater than *bd* the circumference of the circle. The inſcribed polygon is leſs than the circle; and it is alſo leſs than the triangle *abd*, becauſe it is equal to a triangle of a height equal to CQ (which is leſs than CA or *ab*) upon a baſe equal to its perimeter ABDK which is leſs than the circumference of the circle *bd*. Therefore the circle and the triangle *abd* are both conſtantly limits betwixt the external and internal polygons EFGH, ABDK. Let the arch AB be biſected in L, and the tangent at L meet AE, BE, in M and N; and the angle ELM being right, EM muſt be greater than LM or AM, the triangle ELM greater than ALM, EMN greater than the ſum of the triangles ALM, BLN, and conſequently greater than half the ſpace EALB bounded by the tangents EA, EB, and the arch ALB: From which it follows, (by the 1. 10. Eucl. the foundation of this method,) that the circumſcribed polygon may approach to the circle ſo as to exceed it by a leſs quantity than any that can be aſſigned. The inſcribed polygon may alſo approach to the circle ſo as that their difference may become leſs than any aſſignable quantity, as is ſhewn in the Elements. Therefore the circle and the triangle *abd*, which are both limits betwixt theſe polygons, muſt be equal to each other. For, if the triangle *abd* be not equal to the circle, it muſt either be greater or leſs than it. If the triangle *abd* was greater than the circle; then, ſince the external polygon, by increaſing the number of its ſides, might be made to approach to the circle ſo as to exceed it by a quantity leſs than any difference that can be ſuppoſed to be between it and the triangle *abd*; it follows, that the external polygon might become leſs than that triangle, againſt

Fig. 2.

what has been demonſtrated. If the triangle *abd* was leſs than the circle, then the inſcribed polygon, by being made to approach to the circle, might exceed that triangle: which, by what we have ſhewn, is alſo impoſſible.

In general, let any determined quantity AB be always a limit betwixt two variable quantities AP, AQ, which are ſuppoſed to approach continually to it and to each other, ſo that the difference of either from it may become leſs than any aſſignable quantity, or ſo that the ratio of AQ to AP may become leſs than any aſſignable ratio of a greater magnitude to a leſſer. Suppoſe alſo any other determined quantity *ab* to be always a limit betwixt the quantities *ap* and *aq*; and, *aq* being always equal to AQ or leſs than it, let *ap* be either equal to AP or greater than it. Then ſhall



theſe limits AB and *ab* be equal to each other. For, if *ab* was equal to any quantity AE greater than AB, then, by ſuppoſing the ratio of AQ to AP to become leſs than that of AE to AB, the ratio of *aq* to AP would become leſs than the ratio of *ab* to AB: but ſince *aq* is always greater than *ab*, and AP leſs than AB, the ratio of *aq* to AP is always greater than the ratio of *ab* to AB; and theſe being repugnant, it follows, that *ab* is not greater than AB. If *ab* was equal to any quantity A*e* leſs than AB, then, by ſuppoſing the ratio of AQ to AP to become leſs than that of AB to A*e*, the ratio of AQ to *ap* (which, by the ſuppoſition, is either equal to AP, or greater than it) would become leſs than that of AB to *ab*: but ſince AQ always exceeds AB, and *ap* is always leſs than *ab*, the ratio of AQ to *ap* muſt be greater than that of AB to *ab*; and theſe being repugnant, it follows, that *ab* is not leſs than AB. Theſe limits therefore AB, *ab* are equal to each other.

In this manner Archimedes demonſtrates the fourteenth propoſition of the firſt book of his admirable treatiſe concerning
Fig. 3. the ſphere and cylinder. Let DA*ba* be an upright cone; D the

the vertex, C the center and CA the radius of the baſe. Let KI be a mean proportional betwixt DA the ſide of the cone and CA the radius of its baſe; and the convex ſurface of the cone ſhall be equal to a circle of the radius KI. Let LMNS be any external equilateral polygon deſcribed about this circle, *lmns* a ſimilar polygon inſcribed in it; EFGH a ſimilar polygon deſcribed about the baſe of the cone, *efgh* a ſimilar polygon deſcribed in it: then the ſurface of the pyramid DEFGH, which is deſcribed about the convex ſurface of the cone, ſhall be to the polygon EFGH as DA is to CA, or as the ſquare of KI is to the ſquare of CA, or as the polygon LMNS is to the polygon EFGH; and therefore the ſurface of the external pyramid DEFGH, without the baſe, is equal to the polygon LMNS. Let CA, perpendicular to EF, meet *ef* in B; and BR, parallel to DA, meet CD in R: then the polygon *lmns* being to *efgh* as the ſquare of KI is to the ſquare of CA, that is, as DA is to CA, or as BR is to CB; and this being a leſs ratio than that of DB to CB, which is the ſame as the ratio of D*efgh*, the ſurface of the internal pyramid without its baſe, to the internal polygon *efgh*: it follows, that the ſurface of the internal pyramid without its baſe is greater than the polygon *lmns*. The external polygon LMNS is greater than the circle of the radius KI, the internal polygon is leſs than that circle, and the ratio of the external polygon to the internal may become leſs than any aſſignable ratio of a greater quantity to a leſſer. The ſurface of the external pyramid DEFGH, without the baſe, is greater than the convex ſurface of the cone; and the ſurface of the internal pyramid D*efgh*, without the baſe, is leſs than it. Therefore by ſubſtituting, in the general demonſtration of the laſt article, the circle of the radius KI in place of the quantity AB, the external polygon LMNS for AQ, the internal polygon *lmns* for AP, the convex ſurface of the cone for *ab*, the ſurfaces of the external and internal pyramids without their baſes for *aq* and *ap*; it will appear, that the convex ſurface of the cone and the circle of the radius KI are equal to each other. We have given ſo full an account of this demonſtration the rather that a part of it has been ſometimes miſrepreſented.

It is much in the ſame manner that he demonſtrates the thirty firſt, fortieth and forty firſt, or, according to the celebrated Dr. BARROW's numbers, the thirty ſeventh, forty ninth and fiftieth of the ſame treatiſe: which are eſteemed by ſo good a judge, for the uſefulneſs of the propoſitions, the ſubtlety of the invention, and the elegance of the demonſtration, amongſt the moſt valuable diſcoveries in geometry. In the firſt of theſe he demonſtrates, that the ſurface of a ſphere is equal to four times
FIG. 4. the area of a great circle of the ſphere. Let AEBFD*fbe* be an equilateral polygon inſcribed in the generating circle, of a number of ſides that is any multiple of the number four; let GHKLM*lkh* be a ſimilar polygon deſcribed about the circle, having its ſides parallel to thoſe of the internal polygon; and let E*e*, B*b*, F*f* meet the diameter AD in Q, C and N. Suppoſe the ſemicircle ABD, with the inſcribed and circumſcribed polygons, to revolve on the diameter AD: and the conical ſurface generated by the chord AE ſhall be equal to a circle of a radius that is a mean proportional betwixt AE and EQ; the ſurface generated by the chord EB ſhall be equal to a circle of a radius that is a mean proportional betwixt EB (or AE) and the ſum of the perpendiculars EQ and BC; and the ſurface generated by the whole perimeter of the internal polygon equal to a circle of a radius that is a mean proportional betwixt AE and the ſum of the perpendiculars E*e*, B*b* and F*f*. But ſuppoſing B*e* and F*b* to meet the diameter AD in R and S, the triangles DAE, AEQ, Q*e*R, RBC, C*b*S, SFN and DN*f* are ſimilar; and DE is to AE, EQ to AQ, *e*Q to QR, CB to CR, *b*C to CS, FN to SN, and *f*N to DN in the ſame ratio: and therefore the ſum of the perpendiculars E*e*, B*b*, F*f* is to the diameter AD as DE is to AE. From which it follows, that the ſurface generated by the revolution of the internal polygon AEBFD is equal to a circle of a radius that is a mean proportional betwixt AD, the diameter of the generating circle, and DE the right line drawn from one extremity of that diameter to the angle E that is next adjoining to its other extremity. In the ſame manner, the ſurface generated by the perimeter of the external polygon GHKLM is equal to a circle of a radius that is a mean proportional betwixt MH (which is equal to OX or AD) and GM the diameter of

of the circle defcribed about that polygon. Becaufe DE is lefs than AD, but GM greater than AD, it follows that the furface generated by the perimeter of the internal polygon is always lefs, and the furface generated by the perimeter of the external polygon is always greater than a circle of the radius AD which (becaufe AD is double of AC) is equal to four times the area of the generating circle. The furface of the fphere is alfo itfelf always a limit betwixt thefe infcribed and circumfcribed furfaces; and becaufe the ratio of thefe furfaces to each other is the duplicate of the ratio of GH to AE, or of CG to CA, it appears, that by increafing the number of the fides of the polygons the ratio of thefe furfaces may approach nearer to that of equality than any affignable ratio of inequality: therefore by fubftituting, in the general propofition demonftrated above, the furface of the fphere in place of AB, the quadruple area of a great circle in place of *ab*, the furfaces generated by the circumfcribed and infcribed polygons for AQ and AP, and fuppofing *aq* and *ap* refpectively equal to AQ and AP; it will be evident that the furface of the fphere and the quadruple area of its generating circle are equal to each other.

The furface generated by the perimeter AEBF, infcribed in any arch AF, is equal to a circle of a radius that is a mean proportional betwixt DE and AN, which is lefs than AF the mean proportional betwixt AD and AN: and the furface generated by the perimeter GHKL being, for the fame reafon, equal to a circle of a radius that is a mean proportional betwixt AD and GN, which exceeds the chord AF; it follows, that a circle of the radius AF is always a limit betwixt the furfaces generated by the perimeters of the internal and external polygons AEBF and GHKL. But the portion of the fpherical furface generated by the revolution of the arch AF is always a limit betwixt the fame circumfcribed and infcribed furfaces: and thefe being to each other in the duplicate ratio of GH to AE, or of CG to CA, and therefore in a proportion that may approach to the ratio of equality nearer than any ratio of inequality; it follows, that the portion of the fpherical furface generated by the revolution of the arch AF is equal to the area of a circle defcribed with

with a radius equal to the right line AF. The ſurfaces, therefore, generated by arches terminated at A are as the ſquares of their chords, or as their verſed ſines; and parallel planes which divide the diameter of a ſphere in equal parts, divide the ſurface of the ſphere into equal parts at the ſame time.

From theſe propoſitions, it is eaſy to ſee how he was able to compare the ſphere itſelf with the circumſcribed cylinder or inſcribed cone. Let CP, perpendicular from the center on AE, the ſide of the polygon, meet it in P: then the ſolid generated by the revolution of the triangle CAE being equal to a cone of the height CA upon a baſe equal to the circle generated by the perpendicular EQ, and this circle being to the ſurface generated by the revolution of the chord AE as the ſquare of EQ is to the rectangle contained under AE and EQ, that is, as EQ is to AE, or CP to CA; it follows, (ſuch cones being equal as have their heights and baſes reciprocally proportional,) that the ſolid generated by the revolution of the triangle CAE is equal to a cone upon a baſe equal to the ſurface generated by AE of a height equal to CP. Let BE produced meet the axis in T; and, for the like reaſon, the ſolids generated by the triangles CET, CBT ſhall be equal to cones upon baſes reſpectively equal to the ſurfaces generated by the revolution of the right lines ET and BT, of the height CZ equal to CP: ſo that their difference (or the ſolid generated by the triangle CEB) muſt be equal to a cone of the height CP upon a baſe equal to the ſurface generated by the right line EB. In like manner, the ſolid generated by the revolution of the whole inſcribed polygon is equal to a cone of the ſame height CP upon a baſe equal to the ſurface generated by the perimeter of the polygon, which we have ſhewn to be always leſs than four times the area of a great circle of the ſphere. For the ſame reaſon, the ſolid generated by the revolution of the external polygon GHKLM is equal to a cone of the height CO or CA, upon a baſe equal to the ſurface generated by the perimeter of that polygon, which, by what has been demonſtrated, is greater than four times the area of a great circle of the ſphere. A cone, therefore, upon a baſe equal to four times the area of a great circle of the ſphere, of an altitude equal

equal to the radius, is always a limit betwixt the ſolids generated by the external and internal polygons. The ſphere itſelf is alſo a limit betwixt theſe ſolids: and their proportion being the triplicate of that of GH to AE, or of CG to CA, which may approach nearer to a ratio of equality than any aſſignable ratio of inequality; it follows, that the ſphere is equal to ſuch a cone, or to four times a cone that has its baſe equal to the generating circle and its height equal to the radius; and that the ſphere is to the circumſcribed cylinder as two is to three. The ſector of the ſphere generated by the revolution of the ſector of the circle CAE is ſhown, in like manner, to be equal to a cone upon a baſe equal to the portion of the ſpherical ſurface generated by the arch AE (or the circle deſcribed with the radius AE,) of a height equal to CA the radius of the ſphere. Theſe are the thirty ſecond and forty ſecond propoſitions of the ſame excellent treatiſe.

If we ſuppoſe the ſemicircle and ſemiellipſe to revolve on their common axis AB, and to generate a ſphere and ſpheroid; then the cone generated by the triangle AFN ſhall be to the cone generated by the triangle AHN in the duplicate ratio of FN to HN, or of the axis AB to its conjugate; becauſe cones of the ſame height are as their baſes. And in the ſame manner it appears, that the ſolid generated by the revolution of any polygon AFGB, inſcribed in the circle, is to the ſolid generated by the revolution of the correſponding polygon AHKB, inſcribed in the ellipſe, as the ſquare of the axis AB is to the ſquare of its conjugate. The ſolid generated by the triangle TID is to the ſolid generated by the triangle TIE as the ſquare of ID to the ſquare of IE, or as the ſquare of the axis AB to the ſquare of its conjugate: and if any polygon be deſcribed about the circle, and a polygon be alſo deſcribed about the ellipſe, ſo that the points of contact in the circle and ellipſe be always in the ſame perpendiculars to the axis, the ſolids generated by theſe polygons ſhall be to each other in the ſame proportion. By ſuppoſing the circumſcribed and inſcribed polygons in the circle to be equilateral, and to have their correſponding ſides parallel, it may be demonſtrated, that the ſphere is to the ſpheroid in the dupli- FIG. 1.

duplicate ratio of the axis AB to its conjugate, and that the ſolids generated by the ſpaces FNMG, HNMK are in the ſame proportion. The ſpheroid, therefore, is quadruple of the cone of a height equal to half the axis AB, on a baſe equal to the circle deſcribed upon the conjugate axis as its diameter.

As the ſegment of the ellipſe HEK, and the triangle HEK, are of a determined invariable magnitude in a given ellipſe, when the ratio of CP to CE is given; ſo the portion of the ſpheroid (generated by the ellipſe revolving on its axis AB) which is cut off by a plane through HK perpendicular to the plane AEB is of a determined magnitude, when the ratio of CP to CE is given. A cone upon the ſame baſe with this portion, that has its vertex in E, or in *t*, is alſo of an invariable magnitude, when that ratio is given; and there is a general property of the circumſcribed and inſcribed ſolids analogous to that above mentioned of the circumſcribed and inſcribed polygons.

Archimedes takes a different way for comparing the ſpheroid with the cone and cylinder, that is more general, and has a nearer analogy to the modern methods. He ſuppoſes the terms of a progreſſion to increaſe conſtantly by the ſame difference, and demonſtrates ſeveral properties of ſuch a progreſſion relating to the ſum of the terms and the ſum of their ſquares; by which he is able to compare the parabolic conoid, the ſpheroid and hyperbolic conoid, with the cone, and the area of his ſpiral line with the area of the circle. There is an analogy betwixt what he has ſhewn of theſe progreſſions, and the proportions of figures demonſtrated in the elementary geometry; the conſideration of which may illuſtrate his doctrine, and ſerve perhaps to ſhew that it is more regular and compleat in its kind than ſome have imagined *. The relation of the ſum of the terms to the quantity that ariſes by taking the greateſt of them as often as there are terms, is illuſtrated by comparing the triangle with the parallelogram of the ſame height and baſe; and what he has demonſtrated of the ſum of the ſquares of the terms

* See the preface to the *Analyſe des infiniment petits*.

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compared with the ſquare of the greateſt term, may be illuſtrated by the proportion of the pyramid to the priſm, or of the cone to the cylinder, their baſes and heights being equal; and by the ratios of certain fruſtums or portions of theſe ſolids, which may be deduced from the elementary propoſitions.

The baſe AB of the rectangle ABCD being divided into any number of equal parts, AE, EF, FG, GB, and the perpendiculars ET, FV, CR being drawn meeting AC in H, I, L, and CD in T, V and R; the rectangles AH, EI, FL, GC may repreſent a progreſſion of terms that conſtantly increaſe by a common difference equal to the leaſt term AH. The ſum of theſe rectangles form the figure AKHMIPLRCB which is deſcribed about the triangle ACB, and is always greater than the triangle: Their ſum without the greateſt term GC is equal to the figure EHNIQLSB, which is inſcribed in the triangle, and therefore is always leſs than it. The rectangle ABCD is the ſum of as many terms equal to the greateſt GC, as there are terms. And as the triangle ACB, or one half of the rectangle ABCD, is always a limit betwixt theſe circumſcribed and inſcribed figures; ſo, in general, if the greateſt term of any progreſſion of this kind be taken as often as there are terms, then one half of the quantity that is thus produced ſhall be always a limit betwixt the ſum of all the terms and this ſum without the greateſt term: which coincides with the firſt propoſition of the treatiſe of conoids and ſpheroids.

Suppoſe the ſame figure to revolve upon the axis AB, and while the triangle ABC generates a cone equal to the third part of the cylinder deſcribed by the rectangle ABCD, the rectangles AH, EI, FL, GC generate cylinders of equal heights; which are therefore as the ſquares of the right lines EH, FI, GL, BC, that increaſe conſtantly by the ſame difference equal to the firſt term EH. The ſum of theſe cylinders, or the ſolid generated by the figure AKHMIPLRCB circumſcribed about the triangle ABC, always exceeds the cone generated by that triangle: Their ſum without the cylinder generated by the rectangle GC, being equal to the ſolid generated by the inſcribed figure EHNIQLSB, is always leſs than the ſame cone. The cylinder

generated

generated by the rectangle ABCD is the ſum of as many cylinders equal to that generated by GC, as there are terms. And as the cone generated by the triangle ABC, or one third part of the cylinder generated by ABCD, is always a limit betwixt the ſolids generated by the figures AKHMIPLRCB, EHNIQLSB; ſo, in general, one third part of the quantity that is produced by taking the ſquare of the greateſt term of any progreſſion of this kind as often as there are terms, is always a limit betwixt the ſum of the ſquares of all the terms and the ſame ſum without the ſquare of the greateſt term. ARCHIMEDES demonſtrates this propoſition in a general way, and by it the proportion of the area of his firſt ſpiral line to the circumſcribed circle. But PAPPUS * makes uſe of the ratio of the cone to the cylinder for the ſame purpoſe: and ſince we have the example of ſo accurate an author, we ſhall proceed to deſcribe ARCHIMEDES's other theorems of this kind in the ſame manner.

The eleventh propoſition of the treatiſe concerning ſpiral lines appears from comparing the cylinder generated by the rectangle EBCT with the fruſtum of a cone generated by the trapezium EBCH, and with the ſolids generated by the figures EMIPLRCB, EHNIQLSB, the former of which is always greater than that fruſtum, and the latter leſs than it. For the cylinder generated by the rectangle EBCT is to the fruſtum of a cone generated by the trapezium EBCH, as the ſquare of BC is to the rectangle contained under the right lines BC and EH added to one third part of the ſquare of the difference of theſe lines. The demonſtration of which we ſhall ſubjoin, the rather that this propoſition will be of uſe afterwards.

The reſt of the conſtruction remaining as in the fifth figure, FIG. 6. Plate 2. let ES parallel to AC, and HZ parallel to AB, meet BC in S and Z; and, ſuppoſing BX to be a mean proportional betwixt BC and EH, or BZ, complete the parallelogram EBXY. Then let the rectangles EBCT, EBXY, by revolving on the axis AB, generate the cylinders TC*ct*, YX*xy*, and the triangles ABC,

* Collect. Mathem. lib. 4. prop. 21.

AEH,

Fig. 15.

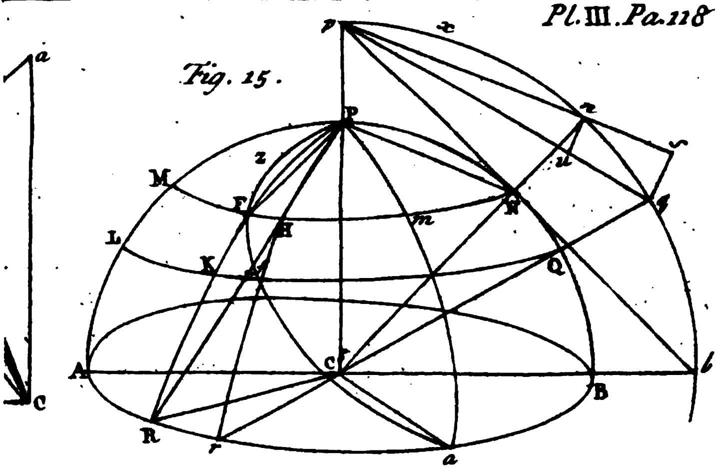
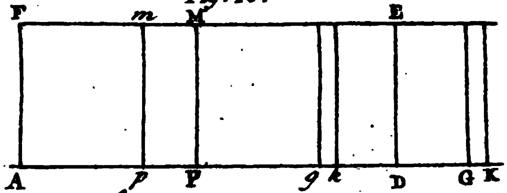
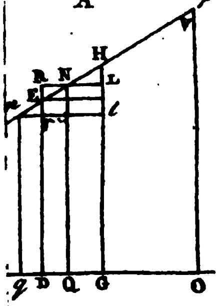


Fig. 16.





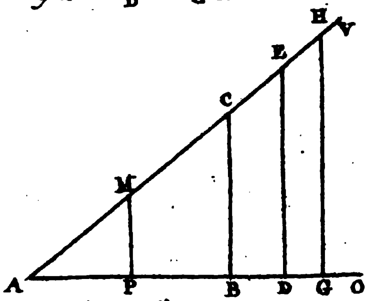
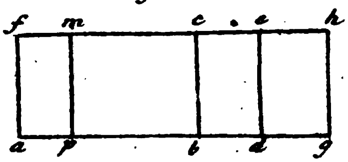
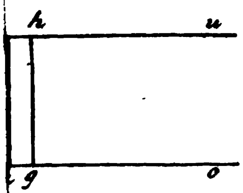


Fig. 18.





AEH, AET, EBC, EBS, EBX, EBZ generate the cones AC*c*, AH*b*, AT*t*, EC*c*, ES*s*, EX*x*, EZ*z*; and the ſolid generated by the parallelogram ESCH ſhall be equal to the cylinder YX*xy*: as may be ſhewn from this known propoſition, That the cube of BC exceeds the two cubes of BZ and CZ (or BS) by three times the parallelepipedon upon the ſquare of BX of the height CZ: From which it follows, that the cone AC*c* exceeds the cones AH*b*, ES*s* by a ſolid equal to the cylinder YX*xy*. The ſame theorem may be demonſtrated from the elementary propoſitions thus: The ſolid generated by the parallelogram ESCH is the ſum of theſe which are generated by the triangles ESC and ECH. That which is generated by ESC is equal to the exceſs of twice the cone EX*x* above the cone EZ*z*, becauſe the ſquare of BC (7. 2. Eucl.) added to the ſquare of BZ is equal to two rectangles under BC and BZ (or twice the ſquare of BX) added to the ſquare of CZ or BS; and therefore the difference of the cones EC*c*, ES*s* (or the ſolid generated by ESC) is equal to the exceſs of twice the cone EX*x* above the cone EZ*z*. The ſolid generated by the triangle ECH is equal to the exceſs of the ſolid generated by AEC (or the cone AT*t*) above the cone AH*b*. But the ſum of the ſquares of BC and BX is to the ſum of the ſquares of BX and BZ as BC is to BZ, or as AB is to AE; and therefore (17. 5. Eucl.) the difference of the ſquares of BC and BZ is to the ſum of the ſquares of BX and BZ as EB is to AE: from which it follows, that the difference of the cones AT*t*, AH*b* (or the ſolid generated by ECH) is equal to the ſum of the cones EX*x* and EZ*z*. Therefore the ſolid generated by the parallelogram ESCH is equal to thrice the cone EX*x*, or to the cylinder XY*yx*; and the fruſtum HC*cb* is equal to that cylinder added to the cone ES*s*: ſo that the cylinder TC*ct* is to the fruſtum HC*cb* as the ſquare of BC is to the ſquare of BX (or the rectangle under BC and EH) added to one third part of the ſquare of BS, or CZ, the difference of BC and EH. It appears, therefore, that the cylinder generated by the rectangle EBCT is to the ſolid generated by the figure EMIPLRCB in a leſs proportion, and to the ſolid generated by the figure EHNIQLSB in a greater proportion, than the ratio of the ſquare of BC to the rectangle under the right

lines BC and EH added to one third part of the ſquare of the difference of theſe right lines. In general, the quantity that is produced by taking the ſquare of the greateſt term (in ſuch a progreſſion as we deſcribed) one time leſs than the number of terms, is to the ſum of the ſquares of the terms without the ſquare of the leaſt of them in a leſs proportion, and to the ſum of their ſquares without the ſquare of the greateſt in a higher proportion, than the ratio of the ſquare of the greateſt term to the rectangle under the extreme terms added to the third part of the ſquare of their difference.

In like manner, the third propoſition of the treatiſe concerning conoids and ſpheroids appears by comparing the ſolid generated by the rectangle HZCT with thoſe generated by the triangle HZC and the figures HMIPLRCZ, NIQLSZ, revolving about the axis AB; ſuppoſing EB to be divided into any equal parts EF, FG, GB. For the ſolid generated by any rectangle NM being to the ſolid generated by the rectangle NL as the difference of the ſquares of FI and EH to the difference of the ſquares of GL and EH, or as the rectangle contained under NI and the ſum of 2EH and NI (6. 2. Eucl.) is to the rectangle under WL and the ſum of 2EH and WL, the ſolids generated by the rectangles NM, WP, ZR increaſe in the ſame proportion as the ſpaces deſcribed in that propoſition, which are ſuppoſed to be applied on a given line, (as 2EH,) and to exceed by ſquares that have their ſides (as NI, WL, ZC) conſtantly increaſing by a difference equal to the ſide of the firſt exceſs. But the ſolid generated by the triangle HZC is equal to the ſolid generated by the rectangle HX added to the cone ES*s*, by what has been demonſtrated; and therefore the ſolid generated by the rectangle HZCT is to the ſolid generated by the triangle HZC as the difference of the ſquares of BC and BZ (or the rectangle CZ*c*) is to the difference of the ſquares of BX and BZ (or the rectangle CZB) added to one third part of the ſquare of BS, or CZ; that is, as Z*c*, or the ſum of BC and BZ, is to the ſum of BZ and one third part of CZ; and therefore as the ſum of 2EH and CZ is to the ſum of EH and one third part of CZ. From which it follows, that the ſolid generated by the

the rectangle HC is to that which is generated by the figure HMIPLBCZ in a less proportion, and to the solid generated by NIQLSZ in a greater proportion, than that ratio. From this and the preceeding theorems ARCHIMEDES deduces many propositions relating to conoids and spheroids, and various portions of these solids: the most considerable of which may be demonstrated in the following manner, that differs little from his own method, and may possibly be the same by which he discovered them.

Let AIC be a parabola, A the vertex, AB a part of the axis, BC an ordinate; complete the rectangle ABCD, and the conoid generated by the parabolic area AICB about the axis AB shall be equal to one half of the cylinder generated by the rectangle ABCD. By continually bisecting the parts of the axis AB let it be divided into the equal parts AE, EF, FG, GB; and let the ordinates EH, FI, GL meet the curve in H, I, L, the right line DC in T, V, R, and the right line AC in X, Y and Z. Complete the rectangles AH, FH, EI, GI, FL, GS, as also the rectangles AX, EY, FZ, GC, E*n*, F*q* and G*s*; and the cylinders generated by AH, EI, FL, GC, having equal altitudes, are as their bases, or as the squares of the ordinates EH, FI, GL BC, and therefore as the parts of the axis AE, AF, AG, AB, or as the rectangles AX, EY, FZ, GC. From which it follows, that the cylinder generated by the rectangle ABCD, the solids generated by the circumscribed figure AKHMIPLRCB and the inscribed figure EHNIQLSB, are in the same proportion to each other as the rectangle ABCD and the figures A*k*X*m*Y*p*ZRCB, EX*n*Y*q*Z*s*B; and consequently one half of the cylinder generated by ABCD is always a limit betwixt the solid circumscribed about the conoid and that which is described in it. The conoid itself is also always a limit betwixt the same circumscribed and inscribed solids; and the difference of these solids being equal to the cylinder generated by the rectangle GC, which, by continually bisecting the parts of the axis, may become less than any solid that can be assigned: it follows, that the conoid generated by the parabolic area AICB is equal to half the cylinder generated by the rectangle ABCD. If a parabolic conoid be cut by planes parallel to each other, but oblique to the axis, the sections are similar. FIG. 7.

ſimilar ellipſes; and any portion of the ſolid cut off by ſuch a plane is ſhewn, in like manner, to be equal to one half of a cylinder of the ſame baſe and altitude.

FIG. 8. Let CA, CB be the ſemiaxes of an ellipſe AIB; complete the rectangle CADB, and join CD. Let CA be divided, by a continual biſection, into the equal parts AE, EF, FG, GC; draw the ordinates EH, FI, GL meeting BD in T, V, R, and CD in X, Y, Z; complete the rectangles AH, FH, EI, GI, FL, CL, and the rectangles DX, VX, TY, RY, VZ, BZ. Then, becauſe the ſquare of any ordinate EH is to the difference of the ſquares of CA and CE as the ſquare of CB is to the ſquare of CA, and the difference of the ſquares of AD (or ET) and EX is to the difference of the ſquares of CA and CE in the ſame proportion; it follows, that the ſquare of EH is equal to the difference of the ſquares of ET and EX, and that the circle deſcribed by EH is equal to the difference of the circles deſcribed by ET and EX: ſo that the cylinder generated by the rectangle AEHK muſt be equal to the ſolid generated by the rectangle DTX*k*. In like manner, the ſolid generated by the circumſcribed figure AKHMIPLRBC is equal to that which is generated by the figure D*k*X*m*Y*p*ZGCB, which always exceeds the ſolid that is generated by the triangle CDB; and the ſolid generated by the inſcribed figure EHNIQLSC is equal to that which is generated by the figure TX*n*Y*q*Z*s*B, which is always leſs than the ſolid generated by that triangle. Therefore the ſolid generated by the triangle CDB and the part of the ſpheroid which is generated by the elliptic area AIBC are conſtantly limits betwixt the ſame circumſcribed and inſcribed ſolids; and the difference of theſe ſolids being equal to the cylinder generated by the rectangle BG, which, by continually biſecting the parts of the axis, may become leſs than any ſolid that can be aſſigned; it appears, that the ſolids generated by the elliptic area AIBC and the triangle DCB are equal, and that the portion of the ſpheroid generated by the elliptic area AEH is equal to the ſolid generated by the triangle DTX revolving about the axis AC. But the ſolid generated by the triangle DX*k*, by what has been demonſtrated, is to the ſolid generated by the rectangle DTX*k* (or the cylinder generated

generated by AEHK) as the ſum of EX and one third part of TX is to the ſum of 2EX and TX, or as the ſum of CA and 2CE is to the triple ſum of CA and CE; and therefore the portion of the ſpheroid generated by the elliptic area AEH is to the inſcribed cone generated by the triangle AEH as the ſum of 2CA and CE is to the ſum of CA and CE.

In the ſame manner, CA being any ſemidiameter of the generating ellipſe, B*b* the conjugate diameter, D*d* a tangent at A, BD and *bd* parallel to CA, ſuppoſe a cylinder BD*db* to be deſcribed about half the ſpheroid AB*kn* touching it in the circumference of the ſection B*kbn* perpendicular to the plane AB*b*, and ſuppoſe the cone CD*rdu* to be inſcribed in this cylinder. Then, if any plane cut BA*b* perpendicularly in the ordinate H*h*, (that meets BD, *bd*, CD, *cd* in T, *t*, X and *x*,) the ſections of the ſpheroid, cylinder and cone made by this plane being ſimilar ellipſes, and the ſquare of EH being equal to the exceſs of the ſquare of ET above the ſquare of EX; it follows, that the ellipſe which is the ſection of the ſpheroid with that plane, is equal to the difference of the ellipſes that are the ſections of the cylinder and cone made by the ſame plane. From which it may be demonſtrated, as in the preceeding article, by continually biſecting the parts of CA, and by circumſcribing and inſcribing cylinders of equal heights about the ſpheroid and cone, that the portion of the ſpheroid AH*h* is equal to the exceſs of the fruſtum of the cylinder DT*td* above the fruſtum of the cone DX*xd*; and that the portion of the ſpheroid AH*h* is to the inſcribed cone AH*h* of the ſame baſe and altitude as the ſum of 2CA and CE is to the ſum of CA and CE. FIG. 9.

Let AIB be an hyperbola, O the center, AC a part of the axis, A the vertex, AD a tangent meeting the aſymptote OD in D, CB an ordinate meeting OD in *e* and D*d* parallel to the axis in *d*; let *ef*, B*b* parallel to the axis meet AD in *f* and *b*, and the reſt of the conſtruction be ſimilar to that of the eighth figure. The ſquare of any ordinate EH being equal to the difference of the ſquares of EX and ET, the circle deſcribed by EH is equal to the difference of the circles deſcribed by EX and ET, FIG. 10.

the

the cylinder generated by the rectangle AEHK equal to the ſolid generated by the rectangle DTXk; and, in like manner, the ſolids generated by the whole circumſcribed and inſcribed figures AKHMIPLRBC, EHNIQLSC are reſpectively equal to the ſolids generated by the figures DkXmYpZged, TXnYqZsd. From which it follows, that the ſolid generated by the triangle Dde is always a limit betwixt the ſolid generated by the figure deſcribed about the hyperbolic area, and that which is generated by the figure deſcribed in it. The portion of the conoid generated by that area is always a limit betwixt the ſame ſolids: and the difference of theſe ſolids being equal to the cylinder generated by the rectangle GB, which, by continuing to biſect the parts of the axis, may become leſs than any ſolid that can be aſſigned; it appears that the portion of the conoid generated by the area ABC is equal to the ſolid generated by the triangle Dde. But this ſolid, by what has been demonſtrated, is to the ſolid generated by the rectangle Ddef, or cylinder generated by the rectangle Cb, as the ſum of 2AD and Ce is to the triple ſum of AD and Ce; and therefore the conoid generated by the hyperbolic area ABC is to the inſcribed cone generated by the triangle ABC as the ſum of 2AD and Ce is to the ſum of AD and Ce, or as the ſum of 2OA and OC is to the ſum of OA and OC. Any portion of the conoid cut off by a plane oblique to the axis, is to the inſcribed cone in a like proportion; as may be demonſtrated much in the ſame manner.

The following general theorem comprehends the preceeding propoſitions and ſeveral others of the ſame kind. It contains a property which extends to all the ſolids that can be generated by any conic ſection revolving upon its axis, including the ſphere and cone, and may be of uſe in menſuration. A fruſtum or portion of any ſuch ſolid terminated by any two parallel planes, and a cylinder of the ſame height with the fruſtum upon a baſe equal to the ſection of the ſolid made by a parallel plane that biſects the altitude of the fruſtum, differ from each other always by the ſame magnitude, in the ſame or in ſimilar ſolids, when the inclination of the planes to the axis and the altitude of the fruſtum are given. In the parabolic conoid this difference vaniſhes,

vanishes, the frustum being always equal to a cylinder of the same height upon the section of the conoid that bisects the altitude of the frustum and is parallel to its bases. In the sphere, the frustum is always less than the cylinder by one fourth part of a right-angled cone of the same height with the frustum, or by one half of a sphere of a diameter equal to that height: and this difference is always the same in all spheres whatsoever, when the altitude of the frustum is given. In the cone, the frustum always exceeds the cylinder by one fourth part of the content of a similar cone that has the same height with the frustum. In the hyperbolic conoid, this excess is the same as in the cone generated by the triangle OC*e* formed by the axis OC FIG. 10. the asymptote O*e* and the perpendicular C*e*, the altitude of the frustums and the inclination of the axis to their bases being the same in both. In the spheroid AB*kb*, the cylinder exceeds the FIG. 9. frustum: and the difference betwixt them is the same as in the cone CD*rd*, the plane D*rd*, or B*kb*, being supposed parallel to those which terminate the frustum. In different inclinations of those planes, when the altitude of the frustum is given, that difference is reciprocally as the cube of the diameter B*b* which is the conjugate of CA, the axis of the frustum. But if the altitude of the frustum be also varied so as to be reciprocally proportional to the diameter B*b*, then the difference betwixt the frustum and cylinder shall be always of the same magnitude in the same spheroid or conoid. When the inclination of the axis of the solid to the planes that terminate the frustum is given, the difference betwixt the frustum and cylinder, in the same or in similar solids, is as the cube of their common altitude.

The truth of this general theorem will easily appear from what has been demonstrated of these solids, if we prove that it obtains in the cone. Resuming, therefore, the construction of the sixth figure, let EB the altitude of the frustum generated by FIG. 11. the trapezium EBCH be bisected in *f*; let *fa* parallel to BC meet AC in *a*, *eb* parallel to AB passing through *a* meet BC and EH in *b* and *e*; and, BR being equal to *b*Z, or one half of CZ, complete the parallelogram EBRV. Then, since CZ is bisected in *b*, the rectangle CBZ (or the square of BX) ad-

ded to the square of bZ, or BR, is equal (6. 2. Eucl.) to the square of Bb; and the cylinder generated by the rectangle Be is therefore equal to the sum of the cylinders generated by the rectangles BY and BV. The frustum of a cone generated by the trapezium EBCH is equal to the cylinder generated by the rectangle BY added to the cone generated by the triangle EBS, by what has been demonstrated; and therefore this frustum exceeds the cylinder generated by the rectangle Be by the excess of the cone generated by the triangle EBS above the cylinder generated by the rectangle BV; that is, (because BR is one half of BS,) by one fourth part of the cone generated by the triangle EBS. But this excess is always of the same magnitude, in the same or in similar cones, when EB the altitude of the frustum is given: Therefore, if a cone is cut by three planes perpendicular to its axis at equal distances from each other, the frustum comprehended betwixt the first and third plane exceeds the cylinder of the same height upon the middle section as its base, by a similar cone of a height equal to the altitude of the frustum; which excess is always of the same magnitude, in the same or in similar cones, when that height is given. In like

FIG. 12. manner, it may be demonstrated, that if the parallel planes Cmc, Knk, Hub, at equal distances from each other, be oblique to the axis of the cone, but perpendicular to the plane $ACBc$ that passes through the axis in the right lines Cc, Kk and Hb; and $ESxs$ be a cone similar to $ACmc$, of the same height with the frustum $CHbc$: then this frustum shall exceed the cylinder of an equal height on the base Knk by one fourth part of the cone $ESxs$.

FIG. 8. The frustum of the spheroid generated by the area EHLG is equal to the solid generated by the trapezium TXZR; and the cylinder generated by the rectangle EMQG is equal to the solid generated by the rectangle $TRqm$, by what has been demonstrated above: Therefore the difference betwixt the frustum and cylinder in the spheroid is the same as in the cone generated by the triangle CAD, the altitudes of the frustums being equal. In the hyperbolic conoid generated by the area

FIG. 10. AIBC revolving on the axis AC, the frustum generated by the area

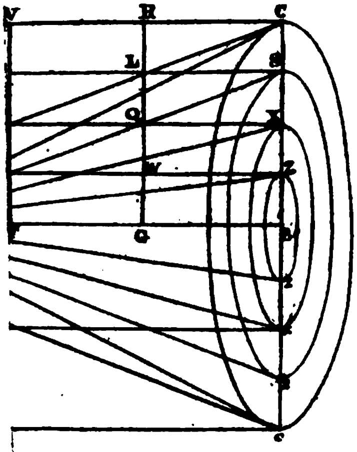
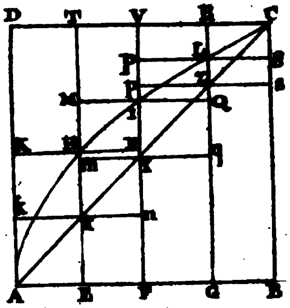


Fig. 7.



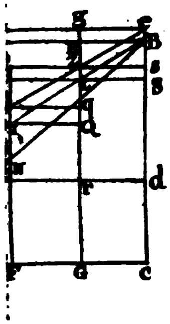


Fig. 8.

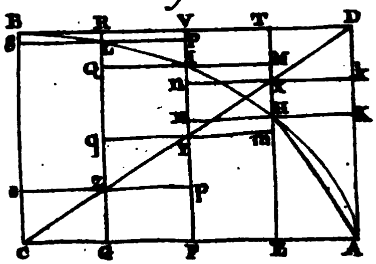


Fig. 11.

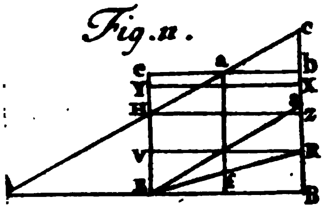
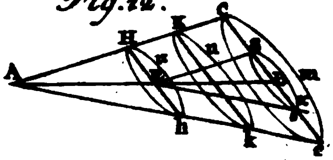


Fig. 12.



area EHLG is equal to the ſolid generated by the trapezium TXZr, and the cylinder generated by the rectangle EMQG is equal to the ſolid generated by the rectangle Trqm; and therefore the difference betwixt rhe fruſtum and cylinder is the ſame as in the cone generated by the triangle OCe revolving on the ſame axis, the altitudes of the fruſtums being equal. And by a ſimilar demonſtratiou this property is extended to any fruſtums of theſe ſolids terminated by parallel planes oblique to the axis of the generating figure.

But to return to ARCHIMEDES: He was the firſt who was able to give the exact quadrature or menſuration of a ſpace bounded by the arch of a curve and a right line, by demonſtrating, that if ABC be any ſegment of a parabola, and DB parallel to the axis of the figure biſecting the baſe AC in D meet the curve in B, the ſegment is to the inſcribed triangle ABC as four is to three, or is equal to the triangle ACa, the right line Ca parallel to the axis being [to BD in that proportion. His demonſtration may be repreſented in the following manner. Let the baſe AC by a continual biſection be divided into the equal parts AR, RG, GS, SD, DX, XE, EY, YC, and let parallels to the axis through the points of diviſion meet the curve in P, H, M, B, N, F, V. Then, if RP paſs through the diviſion of the baſe that is next to the point A, and meet Aa in d, the figure APHMBNFVC inſcribed in the parabolic ſegment ſhall be always equal to the trapezium CRda, as we ſhall ſhew afterwards. By continuing to biſect the parts of the baſe AC, the point R ſhall approach to A, and the trapezium CRda to the triangle CAa ſo that their difference may become leſs than any quantity that can be aſſigned. The inſcribed polygon APHMBNFVC at the ſame time approaches to the area of the parabolic ſegment ABC, ſo that their difference may alſo become leſs than any quantity that is aſſignable; for the tangent at H being parallel to AB, any triangle AHB muſt be greater than half the ſegment AHB in which it is inſcribed. Therefore the parabolic ſegment ABC and the triangle AaC are equal to each other. For if the triangle AaC ſhould be ſuppoſed to exceed the parabolic ſegment ABC by any ſpace, as O; then, ſince by conti-

FIG. 13. Plate 3.

nuing the divisions of the base AC the triangle AR*d* might become less than the space O, the trapezium CR*da*, by approaching to the triangle CA*a* so as to differ from it by a space less than O, might exceed the parabolic segment ABC; and the polygon APHMBNFVC, which is always equal to the trapezium CR*da*, might exceed the parabolic segment in which it is inscribed: which is absurd. If the parabolic segment should be supposed to exceed the triangle AC*a* by any space O; then, since by continuing to bisect the base AC the inscribed polygon might approach to the parabolic segment so as to differ from it by a space less than O, this polygon might exceed the triangle CA*a*. But the inscribed polygon being always equal to the trapezium CR*da*, and the point R being always betwixt A and C, the polygon must be always less than the triangle CA*a*; and these are contradictory. Therefore the parabolic segment ABC and the triangle A*a*C are precisely equal to each other, or the segment ABC is to the inscribed triangle ABC as C*a* is to BD, that is, as four is to three.

We supposed, that the inscribed polygon APHMBNFVC is always equal to the trapezium CR*da*. To complete the demonstration, let DB bisecting AC, GH bisecting AD, RP bisecting AG meet A*a* in the points *b*, *c*, *d* respectively; let GH, RP meet AB, AH in L and *l*; and let HQ parallel to AC meet BD in Q, and P*q* parallel to AB meet GH in *q*. The triangle ABC is equal to the trapezium CD*ba*; because the triangle AC*a* is to the triangle ABC as C*a* is to BD, or as four is to three: and the triangle AC*a* being quadruple of AD*b*, it is to the trapezium CD*ba* in the same proportion of four to three. The sum of the triangles AHB, BFC is equal to the trapezium DG*cb*; because the sum of these triangles is to the triangle ABC as HL (or BQ) is to BD, or as the square of HQ (or GD) is to the square of AD; that is, as one is to four: and the trapezium DG*cb* is to the trapezium CD*ba* (or the triangle ABC) in the same proportion. In like manner, the sum of the triangles APH, HMB, BNF, FVC is equal to the trapezium GR*dc*; because the sum of these triangles is to the sum of the triangles AHB, BFC as P*l* (or H*q*) is to HL, or as the square of P*q* is to the square

ſquare of AL, or as the ſquare of RG is to the ſquare of AG; that is, as one is to four: and the trapezium GR*dc* is to the trapezium DG*cb* (which is equal to the ſum of the triangles AHB, BFC) in the ſame proportion. Thus it appears, that the ſum of the triangles which are ſubducted from the parabolic ſegments, and added to the inſcribed polygon at every new biſection of the parts of the baſe AC, is always equal to the trapezium that is at the ſame time added to CR*da* by biſecting AR the part of the baſe next adjoining to the point A; and therefore the polygon APHMBNFVC inſcribed in the parabolic ſegment is always equal to the trapezium CR*da* terminated by the given right line C*a* and a parallel R*d* drawn through R the diviſion of the baſe next adjoining to the point A. This trapezium CR*da* approaches continually to its limit the triangle CA*a*; the inſcribed polygon at the ſame time approaches to the parabolic ſegment its limit; and theſe are equal to each other, as we have demonſtrated. After all the methods that have been propoſed for demonſtrating the quadrature of the parabola, this, which we have deſcribed from the inventor, ſeems to have a particular elegance. On this occaſion, he ſhews how to find the ſum of any number of terms that decreaſe conſtantly in the proportion of four to one; and, by this example of a geometrical progreſſion, (as it is commonly called,) opened up a ſubject which has been treated at great length by the modern Geometricians.

Archimedes having demonſtrated the quadrature of the parabola; having alſo ſhewn how to approximate to the area of the circle and ellipſe, and how to compare ſpheres, ſpheroids and conoids, or any portions of theſe ſolids, with given cylinders or cones: there ſeems to have been nothing neglected by him that was neceſſary to complete the menſuration of all the figures then received into geometry, and of the ſolids generated by them, if it was not an approximation to the hyperbolic areas. But this does not appear to have been much conſidered by Geometricians, till the analogy of theſe areas with the logarithms was obſerved; and the ſolution of the moſt difficult problems of this kind was found to depend on the meaſures of angles or the a-

reas

reas of the circle, and the meaſures of ratios or the areas of the hyperbola. Beſides what relates to the circle and conic ſections, and the ſolids generated by them, he has given us a treatiſe concerning the ſpiral line which is deſcribed by a point moving with a given velocity along a right line that revolves at the ſame time with an uniform angular motion about one of its extremities. The proportion of the area of this curve to the area of the circumſcribed circle is eaſily deduced from the principles which have been already demonſtrated.

FIG. 14. Let C be the beginning of the ſpiral, CE the ſituation of the revolving ray at the beginning of the motion; and the angles ECA, ACB, BCD, DCE being equal, let CA, CB, CD, CE meet the ſpiral in A, B, D and E: Then, the angular motion of the ray, and the motion of the point that deſcribes the curve along the ray, being both uniform, the right lines CA, CB, CD, CE muſt always increaſe by the ſame difference equal to the firſt line CA. From the center C deſcribe the ſimilar arches AG, BI, DL, EN, which meet CE, CA, CB, CD in G, I, L and N without the ſpiral, as alſo the ſimilar arches AH, BK, DM that are all within the ſpiral, and meet CB, CD, CE in H, K and M. The ſpiral area CABDE is always a limit betwixt the ſum of the circumſcribed ſectors CGA, CIB, CLD, CNE and the ſum of the inſcribed ſectors CAH, CBK, CDM. The circle deſcribed with the radius CE is the ſum of as many ſectors, equal to the greateſt CNE, as there are ſectors; and therefore (ſince the ſectors are as the ſquares of CA, CB, CD, CE) one third part of the area of this circle is alſo a limit betwixt the ſums of the ſame circumſcribed and inſcribed ſectors. The difference of the ſums of theſe ſectors is equal to CNE, which, by biſecting the angles at C, may become leſs than any ſpace that can be aſſigned: From which it follows, that the ſpiral area CABDE is equal to one third part of the circle of the radius CE. In like manner, the whole ſpiral area generated by the ray drawn from the point C to the curve while it makes two revolutions, is the third part of a ſpace that is double of the circle deſcribed with a radius equal to 2CE; and the whole area generated by the ray from the beginning of the motion till after any number of revolutions, is equal

equal to the third part of a ſpace that is the ſame multiple of the circle deſcribed with the greateſt ray, as the number of revolutions is of unit. Any portion of the area of the ſpiral that is terminated by the curve C*m*A and the right line CA is ſhewn, in the ſame manner, to be equal to one third part of the ſector CAG, terminated by the right line CA, and CG the ſituation of the revolving ray when the point that deſcribes the curve ſets out from C. We ſhall afterwards give an account of his theorems concerning the tangents of this curve.

PAPPUS * takes occaſion, from the ſpiral of ARCHIMEDES, to conſider that which is deſcribed by ſimilar motions on the ſurface of a ſphere, and finds a portion of the ſurface terminated by this ſpiral to be equal to the ſquare of the diameter. Let C be the center of the ſphere, ARBA a great circle, P its pole; FIG. 15. and while the quadrant PMA revolves about the pole P with an uniform motion, let a point proceeding from P move with a given velocity along the quadrant, and trace upon the ſpherical ſurface the ſpiral PF*fa*. Let PMA be the ſituation of the quadrant at the beginning of the motion, and let the point that deſcribes the ſpiral come to *a* when the quadrant comes to the ſituation P*ma*. Suppoſe any two quadrants PFR, P*fr* to meet the ſpiral in F, *f* and the circle ARB in R and *r*; through F and *f* from the pole P deſcribe the circles MFN, L*f*Q, meeting the ſemicircle APB in M, N, L and Q; let the circle MFN meet P*fr* in H, and L*f*Q meet PFR in K, and H*f* ſhall be to R*r* as the quadrant PMA is to the arch A*a*, or as the velocity of the point, which deſcribes the ſpiral, along the quadrant PFR to the velocity of R in the circle ARB. Upon CP let C*p* be taken ſo that the ſquare of CP may be to the ſquare of C*p* in the ſame proportion. From C as center deſcribe the circle *pnb* in the ſame plane with APB; produce CN and CQ till they meet this circle in *n* and *q*; from the center *p* deſcribe through *n* and *q* the arches *nu*, *qs* meeting *pq*, *pn* in the points *u* and *s*. Then, ſince the area PFH is to PR*r* as the ſquare of the chord PF is to the ſquare of the chord PR, and the a-

* Collect. Mathem. lib. 4. prop. 30.

rea

rea PR*r* is to the ſector CR*r* as the ſquare of PR is to the ſquare of CR, the area PFH is to CR*r* as the ſquare of PF (or PN) is to the ſquare of CR, or CN. But CR*r* is to CNQ as R*r* is to NQ, and therefore as the ſquare of C*p* is to the ſquare of CP, or as the ſquare of *pn* is to the ſquare of PN; ſo that the area PFH is to the ſector CNQ as the ſquare of *pn* is to the ſquare of CN. The angle *npu* being equal to one half of the angle NCQ, the ſector *pnu* is to one half of the ſector CNQ in the ſame proportion of the ſquare of *pn* to the ſquare of CN: Therefore the area PFH is double of the ſector *pnu*; and, in the ſame manner, the area PK*f* is double of the ſector *pqs*. If we ſuppoſe the quadrants from the pole P to divide the arch A*a* into any number of equal parts, and to meet the ſpiral in any points as F, *f*, the right lines CN, CQ produced to the circle *pnb* will divide the quadrant *pnb* into the ſame number of equal parts: and the ſpaces PFH inſcribed in the area PF*fa*P being always double of the ſectors *pnu* inſcribed in the ſegment *pnbp*, the ſpaces PK*f* that are deſcribed about the ſpiral area being always double of the ſectors *pqs* deſcribed about that ſegment, and the difference betwixt the ſum of the circumſcribed and inſcribed ſectors being equal to the greateſt ſector, (which, by continuing to biſect the parts of the arch A*a*, or quadrant *pnb*, may become leſs than any aſſignable quantity;) therefore the area PF*a*P terminated by the ſpiral PF*a* and the quadrant P*ma* is double of the ſegment *pnbp*. The arch AR*a* is to the quadrant PMA (and the ſector CA*a* to the quadrant CPNB) as the ſquare of C*p* is to the ſquare of CP, or as the area C*pnb* to the area CPNB; and therefore the ſector CA*a* is equal to the quadrant C*pnb*. But it follows, from what has been demonſtrated above after Archimedes, that the ſurface PMAR*am*P is double of the ſector CA*a*; and therefore it is alſo double of the quadrant C*pnb*. From this ſurface take away the part PZF*am*P, and there remains the ſurface PMAR*a*FZP (bounded by the quadrant PMA, the arch AR*a* and the ſpiral PZF*a*) double of the triangle C*pb*, and therefore equal to the ſquare of the right line C*p*.

Suppoſe, for example, the quadrant PMA to make a compleat revolution in the ſame time that the point which traces the ſpiral

ral on the ſurface of the ſphere deſcribes the quadrant, which is the caſe conſidered by PAPPUS. Then, Rr being quadruple of Hf or NQ, Cp muſt be double of CP, and therefore equal to AB the diameter of the ſphere. The portion of the ſpherical ſurface terminated by the whole ſpiral, the circle ARBA and the quadrant PMA, is equal to the ſquare of AB. In any other caſe the area $PMAaFzP$ is to the ſquare of the diameter AB, in the ſame proportion as the arch Aa is to the whole circumference ARBA; and this area is always to the ſpherical triangle PAa, as the inſcribed ſquare is to the circle.

It is demonſtrated in the ſame manner, that the area PZFP is double of the ſegment $pxnp$. But the ſector CAR is to the ſector CPN, as the arch AR is to the Arch PN, and conſequently, as the ſquare of Cp is to the ſquare of CP, or as the ſector cpn is to CPN; ſo that the ſector CAR is equal to cpn: and the ſpherical triangle PAR, being double of the ſector CAR, it is alſo double of Cpn. From this ſpherical triangle ſubduct the area PZFP, and the remainder PMARFZP muſt be double of the triangle Cpn. Therefore the portion of the ſpherical ſurface, terminated by the quadrant PMA, the arches AR, FR and the ſpiral PZF, admits of a perfect quadrature, when the ratio of Cp to CP, or of the arch Aa to the whole circumference, can be aſſigned.

We have now given a ſummary account, of the progreſs that was made by the ancients in meaſuring and comparing curvilineal figures, and of the method by which they demonſtrated all their theorems of this kind. It is often ſaid, that curve lines have been conſidered by them as polygons of an infinite number of ſides. But this principle no where appears in their writings. We never find them reſolving any figure, or ſolid, into infinitely ſmall elements. On the contrary, they ſeem to avoid ſuch ſuppoſitions, as if they judged them unfit to be received into geometry, when it was obvious that their demonſtrations might have been ſometimes abridged by admitting them. They conſidered curvilineal areas as the limits of circumſcribed or inſcribed figures of a more ſimple kind, which approach to theſe

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limits, (by a biſection of lines, or angles, that is continued at pleaſure,) ſo that the difference betwixt them may become leſs than any given quantity. The inſcribed or circumſcribed figures were always conceived to be of a magnitude and number that is aſſignable; and from what had been ſhewn of theſe figures, they demonſtrated the menſuration, or the proportions, of the curvilineal limits themſelves, by arguments *ab abſurdo*. They had made frequent uſe of demonſtrations of this kind from the beginning of the Elements; and theſe are in a particular manner adapted for making a tranſition from right-lined figures to ſuch as are bounded by curve lines. By admitting them only, they eſtabliſhed the more difficult and ſublime part of their geometry on the ſame foundation as the firſt elements of the ſcience. Nor could they have propoſed to themſelves a more perfect model.

We have already obſerved, how ſolicitous ARCHIMEDES appears to be, that his demonſtrations ſhould be found to depend on thoſe principles only that had been univerſally received before his time. In his treatiſe of the quadrature of the parabola, he treats of a progreſſion whoſe terms decreaſe conſtantly in the proportion of four to one, which we expreſſed by the trapezia
FIG. 13. CD*ba*, DG*cb*, GR*dc*, &c. But he does not ſuppoſe this progreſſion to be continued to infinity, or mention the ſum of an infinite number of terms; though it is manifeſt, that all which can be underſtood by thoſe who aſſign that ſum was fully known to him. He appears to have been more fond of preſerving to the ſcience all its accuracy and evidence, than of advancing paradoxes; and contents himſelf with demonſtrating this plain property of ſuch a progreſſion, That the ſum of the terms continued at pleaſure, added to the third part of the laſt term, amounts always to four thirds of the firſt term; as the ſum of the trapezia CD*ba*, DG*cb*, GR*dc*, added to one third part of the laſt trapezium GR*dc*, amounts always to four thirds of the firſt trapezium CD*ba*, or to the triangle CA*a*. Nor does he ſuppoſe the chords of the curve to be biſected to infinity; ſo that after an infinite biſection the inſcribed polygon might be ſaid to coincide with the parabola. Theſe ſuppoſitions had been new

new to the Geometricians in his time, and ſuch he appears to have carefully avoided.

He has demonſtrated many other theorems of this kind from the properties of certain progreſſions, the terms of which correſpond to the circumſcribed and inſcribed figures: but he never ſuppoſed theſe terms, or figures, to increaſe or decreaſe by infinitely ſmall differences, and to become infinite in number; that their ſum might be ſuppoſed equal to the curvilineal area, or ſolid. It was ſufficient for his purpoſe, to aſſign a quantity that is always a limit betwixt the ſum of all the terms of the progreſſion, and the ſame ſum without one of the extreme terms; as the area, or ſolid, is always a limit between the ſum of the circumſcribed and the ſum of the inſcribed figures, which ſums differ from each other in the extreme figures only. He conſidered but one decreaſing geometrical progreſſion, and ſhewed how to find in an arithmetical progreſſion the ſum of the terms and of their ſquares only. Of late, other geometrical progreſſions have been employed ſucceſsfully, for meaſuring the areas of curves; the ſums of the cubes, and of all the other powers, of the terms in an arithmetical progreſſion have ſerved for the ſame purpoſe: and each of his diſcoveries has produced ſome extenſive theory in the modern geometry.

His method has been often repreſented as very perplexed, and ſometimes as hardly intelligible. But this is not a juſt character of his writings, and the ancients had a different opinion of them *. He finds it neceſſary indeed to premiſe ſeveral propoſitions to the demonſtration of the principal theorems; and on this account his method has been excepted againſt as tedious. But the number of ſteps is not the greateſt fault a demonſtration may have; nor is this number to be always computed from thoſe that may be propoſed in it, but from thoſe that are neceſ-

* Plutarch celebrates the ſimplicity and plainneſs with which he treats the moſt difficult and abſtruſe queſtions: Ου γάρ ἐστιν ἐν γεωμετρίᾳ χαλεπωτέρας ϗ βαρυτέρας ὑποθέσεις ἐν ἁπλουστέροις λαβεῖν ϗ καθαρωτέροις στοιχείοις γραφομένας. *Plut. in vita Marcelli.*

fary to make it full and conclufive. Befides, thefe preliminary propofitions are generally valuable on their own account, and render our view of the whole fubject more clear and compleat. In his treatife of the fphere and cylinder, for example, by his demonftrating fo fully the menfuration of the furfaces and folids, generated by the internal and external polygons, we not only fee how the furface and folid content of the fphere itfelf is determined, but we acquire a more perfect knowledge of this theory, and of all that relates to it, with a fatisfaction that we are fenfible is often wanting in the incompleat demonftrations of fome other methods.

By fo many valuable difcoveries demonftrated in fo accurate a manner, and by the admirable ufe he made of his knowledge in the celebrated fiege of his native city *, and upon other occafions †, ARCHIMEDES has diftinguifhed himfelf amongft the Geometricians, and has done the greateft honour to this part of learning. He has not however efcaped the cenfures of fome writers, who being unskilful in geometry, and unable to reconcile their own conceits with his demonftrations, have reprefented him as in an error, and mifleading Mathematicians by his authority ||. But though Mathematicians may be grateful, authority has not any place in this fcience; and no Geometrician ever pretended, from the higheft veneration for ARCHIMEDES, Sir ISAAC NEWTON or others, to reft on their judgment

* How he difconcerted all the efforts of two Roman armies, commanded by the Proconful Marcellus and by Appius Claudius, in the fiege of Syracufe, (till the city being taken by furprife and treachery, an end was put to his life and enquiries at once,) is defcribed at length by Polybius, Livy, Plutarch, &c. He was called πολυμήχανος and ἑκατόγχειρ; and, according to Plutarch, acquired the reputation of more than human learning. Medals of Syracufe, with figures that are fuppofed to refer to his difcoveries, ferve rather to juftify his countrymen from the reproach of ingratitude which fome have imputed to them, than to do honour to the immortal Archimedes. *Paruta Sicil. Spanhem. in orat. 1. Juliani.*

† Diodorus Siculus tells us, (*lib.* 5.) that when Archimedes travelled into Egypt, he invented machines that were of great ufe to that nation, and procured him an univerfal reputation.

|| Decepit illos auctoritas Archimedis, &c. *Hobbes de principiis & ratiocinatione Geometrarum.* The learned Jofeph Scaliger and others have alfo writ againft him.

ment in a matter of geometrical demonſtration. The purſuit of general and eaſy methods may have induced ſome to make uſe of exceptionable principles; and the vaſt extent, which the ſcience has of late acquired, may have occaſioned their propoſing incompleat demonſtrations. They may have alſo ſometimes fallen into miſtakes: but it will be found difficult to aſſign one falſe propoſition that has been ever generally received by Geometricians; and it is hardly poſſible, that accuſations of this nature can be more miſplaced.

In what Archimedes had demonſtrated of the limits of figures and progreſſions, there were valuable hints towards a general method of conſidering curvilineal figures; ſo as to ſubject them to menſuration by an exact quadrature, an approximation, or by comparing them with others of a more ſimple kind. Such methods have been propoſed of late in various forms, and upon different principles. The firſt eſſays were deduced from a careful attention to his ſteps *. But, that his method might be more eaſily extended, its old foundation was abandoned, and ſuppoſitions were propoſed which he had avoided. It was thought unneceſſary to conceive the figures circumſcribed or inſcribed in the curvilineal area, or ſolid, as being always aſſignable and finite; and the precautions of Archimedes came to be conſidered as a check upon Geometricians, that ſerved only to retard their progreſs. Therefore, inſtead of his aſſignable finite figures, indiviſible or infinitely ſmall elements were ſubſtituted; and theſe being imagined indefinite, or infinite, in number, their ſum was ſuppoſed to coincide with the curvilineal area, or ſolid.

It was however with caution that theſe ſuppoſitions were at firſt employed in geometry by Cavalerius, the ingenious author of the method of indiviſibles, and by others. He diſcovered a method, which he found to be of a very extenſive uſe,

* C'etoit en obſervant de prês la marche d'Archimede qu'il [*M. de Roberval*] etoit arrivé à cette ſublime & merveilleuſe ſcience, *&c.* *Ouvrag. de l'Acad. Royal.* 1693. This is generally acknowledged by the writers of that time.

and

and of an eaſy application, for meaſuring or comparing planes and ſolids; and would not deprive the public of ſo valuable an invention. In propoſing it, he ſtrove to avoid * the ſuppoſing magnitude to conſiſt of indiviſible parts, and to abſtract from the contemplation of infinity; but he acknowledged, that there remained ſome difficulties in this matter which he was not able to reſolve. Therefore he ſubjoined more unexceptionable demonſtrations to thoſe he had deduced from his own principles; and the diſputes which enſued (the firſt of any moment that were known between Geometricians) juſtified his precautions. Afterwards, infinitely ſmall elements were ſubſtituted in place of his indiviſibles; and various improvements were made in this doctrine. The method of ARCHIMEDES, however, was often kept in view, and frequently appealed to as the ſureſt teſt of every new invention. The harmony betwixt the concluſions that aroſe from the old and new methods contributed not a little to the credit which the latter at firſt acquired; till being more and more reliſhed, they came at length to be generally admitted on their own evidence, and ſeem'd to merit ſo favourable a reception, by the great advantages that were derived from them for reſolving the moſt difficult problems, and demonſtrating the moſt general theories, in a brief and eaſy manner.

But when the principles and ſtrict method of the ancients, which had hitherto preſerved the evidence of this ſcience entire, were ſo far abandoned, it was difficult for the Geometricians to determine where they ſhould ſtop. After they had indulged themſelves in admitting quantities, of various kinds, that were not aſſignable, in ſuppoſing ſuch things to be done as could not poſſibly be effected, (againſt the conſtant practice of the ancients,) and had involved themſelves in the mazes of infinity; it was not eaſy for them to avoid perplexity, and ſometimes error,

* Quoad continui compoſitionem, manifeſtum eſt ex præoſtenſis, ad ipſum ex indiviſibilibus componendum nos minimè cogi: ſolùm enim continua ſequi indiviſibilium proportionem, & è converſo, probare intentum fuit; quod quidem cum utraque poſitione ſtare poteſt. Tandem verò dicta indiviſibilium aggregata non ita pertractavimus, ut infinitatis rationem propter infinitas lineas ſeu plana ſubire videantur, *&c. Cavalerii Geom. indiviſ. lib. 7. præf.*

or

or to fix bounds to theſe liberties when they were once introduced. Curves were not only conſidered as polygons of an infinite number of infinitely little ſides, and their differences deduced from the different angles that were ſuppoſed to be formed by theſe ſides; but infinites and infiniteſimals were admitted of infinite orders, every operation in geometry and arithmetic applied to them with the ſame freedom as to finite real quantities, and ſuppoſitions of this nature multipled, till the higher parts of geometry (as they were moſt commonly deſcribed) appeared full of myſteries.

From geometry the infinites and infiniteſimals paſſed into philoſophy, carrying with them the obſcurity and perplexity that cannot fail to accompany them. An actual diviſion, as well as a diviſibility of matter *in infinitum*, is admitted by ſome. Fluids are imagined conſiſting of infinitely ſmall particles, which are compoſed themſelves of others infinitely leſs; and this ſubdiviſion is ſuppoſed to be continued without end. Vortices are propoſed, for ſolving the phænomena of nature, of indefinite or infinite degrees, in imitation of the infiniteſimals in geometry; that, when any higher order is found inſufficient for this purpoſe, or attended with an inſuperable difficulty, a lower order may preſerve ſo favourite a ſcheme. Nature is confined in her operations to act by infinitely ſmall ſteps. Bodies of a perfect hardneſs are rejected, and the old doctrine of atoms treated as imaginary, becauſe in their actions and colliſions they might paſs at once from motion to reſt, or from reſt to motion, in violation of this law. Thus the doctrine of infinites is interwoven with our ſpeculations in geometry and nature. Suppoſitions, that were propoſed at firſt diffidently, as of uſe for diſcovering new theorems in this ſcience with the greater facility, and were ſuffered only on that account, have been indulged, till it has become crowded with objects of an abſtruſe nature, which tend to perplex it and the other ſciences that have a dependence upon it.

They who have made uſe of infinites and infiniteſimals with the greateſt liberty, have not agreed as to the truth and reality they would aſcribe to them. The celebrated Mr. LEIBNITZ owns

owns them to be no more than fictions. Others place them on a level with finite quantities, and endeavour to demonstrate their reality, from magnitude's being susceptible of augmentation and diminution without end, from the properties of the progressions of numbers that may be continued at pleasure, and from the infinity which some Geometricians have ascribed to the hyperbolic area. But in these arguments they seem to suppose the infinity which they would demonstrate.

It was a principle of the ancient Geometricians, That any given line may be produced, and its parts subdivided, at pleasure: but they never supposed it to be produced, till it should become infinitely great; or to be subdivided, till its parts should become infinitely small. It does not necessarily follow, that, because any given right line may be continued further, it can be produced till it become actually infinite, or that we are able to conceive such a line to be described, so as to admit it in geometry. In general, magnitude is capable of being increased without end; that is, no term or limit can be assigned or supposed beyond which it may not be conceived to be further increased. But from this it cannot be inferred, that we are able to conceive or suppose magnitude to be really infinite *: or, if we are

* In a late treatise ascribed to a celebrated author, justly esteemed for his various writings, several arguments are proposed, for admitting magnitude actually infinite: not that kind which has no limits, comprehends all, and can receive no addition, which he calls *metaphysical*; but that which he defines to be greater than any finite magnitude, which he distinguishes from the former, and calls *geometrical*. *Puisque la grandeur est susceptible d'augmentation sans fin on la peut concevoir ou supposer augmentée une infinité des fois, c'est-à-dire qu'elle sera devenuë infinie. Et, en effet, il est impossible que la grandeur susceptible d'augmentation sans fin soit dans le même cas que si elle n'en etoit pas susceptible sans fin. Or, si elle ne l'etoit pas, elle demeureroit toujours finie; donc etant susceptible d'augmentation sans fin, elle peut ne demeurer pas toujours finie, ou, ce qui est le même, devenir infinie.* Elem. de la geom. de l'infini, § 83. Because magnitude is susceptible of augmentation without end, the author concludes, that we may suppose it augmented an infinite number of times. But, by being susceptible of augmentation without end, we understand only, that no magnitude can be assigned or conceived so great but it may be supposed to receive further augmentation, and that a greater than it may still be assigned or conceived. We easily conceive that a finite magnitude may become greater and greater without end, or that no termination

are able to join infinity to any supposed idea of a determinate quantity, and to reason concerning magnitude actually infinite, it is not surely with that perspicuity that is required in geometry.

mination or limit can be assigned of the increase which it may admit: but we do not therefore clearly conceive magnitude increased an infinite number of times. Mr. Lock acknowledges, that we easily form an idea of the infinity of number, to the end of whose addition there is no approach: but he distinguishes betwixt this and the idea of an infinite number; and subjoins, that how clear soever our idea of the infinity of number may be, there is nothing more evident than the absurdity of the actual idea of an infinite number.

The latter part of the argument amounts to this: "It is impossible that magnitude being susceptible of augmentation without end, can be in the same case "as if it was not susceptible of augmentation without end. But if it was not "susceptible of augmentation without end, it would remain always finite. There"fore, since it is susceptible of augmentation without end, the contrary must be "allowed; that is, it may not always remain finite, or it may become infinite." The Force of which argument seems to be taken off, by considering, that, if magnitude was not susceptible of augmentation without end, it would not only remain always finite, but there would necessarily be a term, limit or degree of magnitude which could never be exceeded, or there might be a greatest magnitude. And, by allowing that there is no such term or limit, magnitude is not supposed to be in the same case as if it was not susceptible of augmentation without end, though we should refuse that it may become infinite. What is opposite to the supposing magnitude susceptible of augmentation without end, is not the supposing it always finite, (for finite magnitude is capable of being increased without end;) but the supposing it susceptible of no augmentation at all, or of an augmentation that has a limit or end.

The series of numbers, 1, 2, 3, 4, &c. in their natural order, may be continued without end; and it is said, that "we never come nearer the end of the "progression, how great soever the number may be to which we arrive; which "is a character that cannot belong to a series of a finite number of terms. There"fore this natural series has an infinite number of terms." And it is added, that "though we can go over a finite number of terms only, yet all the terms of this "infinite progression are equally real." But if we can conceive this series to have any end, it seems to be evident, that we must approach to this end as we proceed from the beginning towards it; and that, while we advance, the distance of any term from the end must decrease (whether this distance be called finite or infinite) by the same quantity as the distance from any subsequent term decreases, or the distance from the beginning of the series increases. If we cannot conceive the series to have an end, then we can have no idea of its last term. If we suppose this series to be continued to infinity, it would indeed be absurd, after such a supposition, to say that the number of its terms is finite. But, in treating this science strictly, it may perhaps be better to avoid this supposition. For if it is only a finite number of terms we can clearly conceive, how shall we judge of the reality of the rest? or wherein shall we place the reality of those which it is impossible for us to assign? of which two kinds are said to be in this same series,

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try. In the same manner, no magnitude can be conceived so small, but a less than it may be supposed; but we are not therefore able to conceive a quantity infinitely small. A given magnitude

each infinite in number; the first of which are said to be finite, but indeterminable; the latter, actually infinite.

The argument from the infinity of the hyperbolic area is much insisted on. "The hyperbolic area (*Elem. de la geom. de l'infin. pref.*) is as really infinite, as "a determined parabolic area is two thirds of the circumscribed parallelogram. "It is trifling to say, that the one can be actually described, and the other cannot. Geometry is entirely intellectual, and independent of the actual description and existence of the figures whose properties it discovers. All that is "conceived necessary in it has the reality which it supposes in its object. Therefore the infinite which it demonstrates is as real as that which is finite, *&c.*" And the learned author, after insisting on this subject, concludes, that, "not to "receive infinity as it is here represented, with all its necessary consequences, is "to reject a geometrical demonstration; and that he who rejects one, ought to "reject them all." But though the actual description of the figures which are considered in geometry be not necessary, yet it is requisite that we should be able clearly to conceive that they may exist; and a distinct idea of the manner how they may be supposed to be described or generated is necessary, that they may have a place in this science. Principles that are proposed as of the most extensive use, and as the foundation of all the sublime geometry, ought to be clear and unexceptionable. If this science is entirely intellectual, or if the reality of its objects is to be considered as having a dependence on their being conceived by the mind, it would seem that there must be a difference betwixt the reality of finite assignable lines or numbers, and the reality we can ascribe to infinite lines or numbers, which are not assignable, and cannot be supposed to be produced or generated but in a manner that is allowed to be inconceivable. As for what is said of the parabolic and hyperbolic areas, we can conceive any portion of the parabola to be accurately described, and its area to be determined, though no exact figure of this kind should ever exist. We can also conceive, that the hyperbola and its asymptote may be produced to any assignable distance: but we do not so clearly conceive that they may be produced to a distance greater than what is assignable; and we may well be allowed to hesitate at such a supposition in strict geometry. Any finite space being proposed, the hyperbolic area (terminated by the curve, the asymptote and a given ordinate) will exceed it by producing the curve and asymptote to an assignable distance; and there is no assignable limit in this (as in some other cases) which the area may not surpass in magnitude. Therefore it is said, that this area would be infinite, if the curve and asymptote could be infinitely produced. But no argument for admitting magnitude actually infinite can be deduced from this, which does not more easily appear from hence, that a parallelogram of a given height would be infinite if it could have an infinite base: from which it cannot be inferred that such a base or parallelogram can actually exist. It is often said, that a rectangle of a given height on an imaginary base (as the Analysts speak) is imaginary: but we cannot thence infer, that an imaginary line or rectangle can exist. It is not however

gnitude may be supposed to be divided into any assignable number of parts; but it cannot therefore be conceived to be divided into a number of parts greater than what is assignable. The parts

ever our intention to maintain the impossibility of infinite magnitude; but to shew, that such doctrines are not necessary consequences of the received principles of this science, and not very proper to be admitted as the ground-work of the high geometry.

As for the hyperbolic areas of a higher kind, which are said to be of a finite magnitude though infinitely produced, the meaning is, that there is a certain finite space which such an area never can equal, though the curve and its asymptote be produced never so far; to which however the area approaches, so that the excess of that finite space above it may become less than any space that may be proposed, by producing the curve and its asymptote to a distance that is assignable: As, the sum of the trapezia CD*ba*, DG*cb*, GR*dc*, &c. that are determined by bisecting AR continually, is always less than the triangle AC*a*, but approaches to it so that their difference AR*d* may become less than any given space O: Or, as the sum of the right lines CD, DG, GR, &c. is always less than CA, but approaches to it, so that, by continuing the bisection, the difference AR may become less than any assigned quantity. But we shall have occasion to treat of these afterwards more fully. FIG. 13.

In the same treatise (§ 196.) a proof is offered, to shew, that, in the infinite series of numbers proceeding in their natural order, there are finite numbers whose squares become infinite, which are called indeterminable, and are supposed to occupy the obscure passage from the numbers that are assignable to those that are infinite. A greatest finite square is supposed in this progression, and represented by *nn*; all that preceed it are finite, and all that follow after it are supposed infinite. The numbers in this progression between *n* and *nn*, being less than *nn*, are finite; but being greater than *n*, their squares are greater than *nn*, and therefore, by the supposition, are infinite. But how can we admit the supposition of a greatest finite square number, such as is here expressed by *nn*? The number *nn*, being finite, is not the next to it in the progression, (which exceeds it by unit only,) also finite. Should we allow, that a finite number becomes infinite by adding unit to it, or even by squaring it, how shall we distinguish finite from infinite? We commonly conceive finite magnitude to be assignable, or to be limited by such as are assignable, and to be susceptible of further augmentation: and therefore infinite magnitude would seem to imply, either that which exceeds all assignable magnitude, or that which cannot admit of any further augmentation; these being directly opposite to what we most clearly conceive of finite magnitude. But neither of these constitute the idea of infinite magnitude, as it must be understood in that treatise. The former is applicable to those numbers which the author calls finite and indeterminable; which, being supposed to produce infinite squares, must therefore exceed all assignable numbers whose squares are assignable and finite. The latter is ascribed to that infinite only which he calls metaphysical, and excludes from geometry. We are at a loss to form a distinct idea even of finite itself as it is here understood; and it would seem, that

parts of a given line may be supposed to be continually bisected till they become less than any line that is proposed; and this is sufficient for completing the demonstrations of the ancients. But

that the more art and ingenuity is employed in penetrating into the doctrine of infinites, it becomes the more abstruse.

A proof is offered *à posteriori* (§ 393.) to shew, that there are finite fractions in the series 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, *&c.* whose squares become infinitely little in the series 1, $\frac{1}{4}$, $\frac{1}{9}$, $\frac{1}{16}$, $\frac{1}{25}$, *&c.* The sum of the first series corresponds with the area included betwixt the common hyperbola and its asymptote; and is said to be infinite when the series is supposed to be continued to infinity. The sum of the latter series corresponds with the area of an hyperbola of a higher order, and is said to be finite, even when the series is supposed to be continued to infinity; because there is a limit which this sum can never equal, to which however it continually approaches, as we have already described. This being allowed, it is supposed farther, that there is an infinite number of finite terms in the first progression; and it is thence demonstrated, that there are finite fractions in the first series whose squares become infinitely little in the second, thus: " If it should be pretended, " that all the finite terms in the first series have their squares finite in the second, " there would be an infinite number of finite terms in the second as well as in " the first; and the sums of both would be infinite: so that the contrary of an " undoubted truth, that is universally received, would be demonstrated." If we could allow that there is an infinite number of finite terms in the first series, this argument might have some weight. But this is a supposition we cannot admit. For the denominator of any fraction in the first series is always equal to the number of terms from the beginning, and must be supposed infinite when the number of terms is supposed infinite; but a fraction that has unit for its numerator, and is supposed to have an infinite number for its denominator, cannot be supposed finite, but infinitely little: so that we cannot suppose an infinite number of terms in the first series to be finite. It is often said in this treatise, that there is an infinite number of finite terms in the natural series 1, 2, 3, 4, 5, *&c.* continued to infinity. But we are at a loss to conceive how this can be admitted; since, in any such progression, the last or greatest term is always equal to the number of terms from the beginning, and cannot be supposed finite when the number of terms is supposed infinite. There is an assignable limit which the sum of the terms of the second series never amounts to; but there is no assignable limit which the sum of the first series may not surpass, (as we shall shew afterwards;) and the sum of the terms of the first is greater than the sum of the corresponding terms of the second, in a ratio that by continuing the terms may exceed any assignable ratio of a greater magnitude to a lesser: and as this is easily understood and demonstrated, so there is no necessity for having recourse to such abstruse principles in order to account for it.

It is of no use to cite authorities on this subject, but as they may justify us in establishing so noble a part of geometry for avoiding principles that are so much contested. What Aristotle taught of infinite magnitude is well known. Mr. Leibnitz, who for obvious reasons cannot be suspected of any prejudice against the doctrine of infinites, expresses himself thus: *On s'embarasse dans les series des*

But it is acknowledged by those who have treated the doctrine of infinites in the fullest manner, that "there is something in-"conceivable in supposing an infinitely great or infinitely small "number or figure to be produced or generated; and that the "passage from finite to infinite is obscure and incomprehen-"sible:" and therefore it is better for us, in treating of so strict a science as geometry, to abstract from these suppositions. The abstruse consequences, that have been deduced from them by ingenious men, may the rather induce us to beware of admitting them as necessary principles in this science, and to adhere to its ancient principles.

Mr. Lock, who wrote his excellent essay, "that we might "discover how far the powers of the understanding reach, to

des nombres qui vont à l'infini. On conçoit un dernier terme, un nombre infini, ou infiniment petit; mais tout cela ne sont que des fictions. Tout nombre est fini & assignable, toute ligne l'est de même. Essai de Theodicée, disc. prelim. § 70.

We have subjoined these remarks, at the desire of some persons for whom we have a great regard, to shew why we have not followed an author who has merited so well of Mathematicians, and who on every other occasion has been justly applauded for his clear and distinct way of explaining the abstruse geometry. They who treated of infinites before him proceeded, as he observes, with a timorousness which the contemplation of such an object naturally inspires: *Quand on y etoit arrivé,* (says he) *on s'arrestoit avec une espece d'effroy & de sainte horreur — on regardoit l'infini comme un mistere qu'il falloit respecter, & qu'il n'etoit pas permis d'approfondir.* They stopt when they came to infinity with a sort of holy dread, and respected it as an incomprehensible mystery. He adventures farther, in order to discover the source, and penetrate into the first principles of geometrical truth. Infinity, according to him, is the great trunk from which its various branches are derived, and to which they all lead. In this great pursuit he displays infinite and finite with a freedom that puts us in mind of the ancient Poet and his Gods, whom he represents with the passions of men, and mingles in their battles. We doubt not, that if a full and perfect account of all that is most profound in the high geometry could have been deduced from the doctrine of infinites, it might have been expected from this author: But our ideas of infinites are too obscure and unadequate to answer this end; and there are many things advanced by all those who have applied them with great freedom in geometry, that give ground to a remark like to Mr. de St. Evremond's, when he observes, that "it is surprising to find the ancient Poets so scrupulous to pre-"serve probability in actions purely human, and so ready to violate it in repre-"senting the actions of the Gods." Some have not only admitted infinites and infinitesimals of infinite orders, but have distinguished even nothings into various kinds: and if such liberties continue, it is not easy to foresee what absurdities may be advanced as discoveries in what is called the sublime geometry.

"what

" what things they are in any degree proportionate, and where " they fail us," obſerves, " that whilſt men talk and diſpute of " infinite magnitudes, as if they had as compleat and poſitive " ideas of them as they have of the names they uſe for them, " or as they have of a yard, or an hour, or any other determi- " nate quantity, it is no wonder if the incomprehenſible nature " of the thing they diſcourſe of, or reaſon about, leads them " into perplexities and contradictions; and their minds be over- " laid by an object too large and mighty to be ſurveyed and ma- " naged by them." Mathematicians indeed abridge their computations by the ſuppoſition of infinites; but when they pretend to treat them on a level with finite quantities, they are ſometimes led into ſuch doctrines as verify the obſervation of this judicious author. To mention an inſtance or two: The progreſſion of the numbers 1, 2, 3, 4, 5, *&c.* in their natural order, is ſuppoſed to be continued to infinity, till by the continual addition of units an infinite number is produced, which is conceived to be the termination of this ſeries. This infinite number is ſuppoſed to be ſtill capable of augmentation and diminution; and yet it is ſaid, " that it is neither increaſed nor diminiſhed " by the addition or ſubtraction of the ſame units from which " it was ſuppoſed to be generated." In a progreſſion of this kind, the number of terms is always equal to the laſt or greateſt term, and is finite when the laſt term is finite. If the number of terms be ſuppoſed infinite, the laſt term cannot be finite; and yet it is ſaid, " that in ſuch a progreſſion continued to in- " finity there is an infinite number of finite terms." It is evident, that no finite number can become infinite by the addition of unit or of any other finite number; and yet " a greateſt fi- " nite ſquare number is ſuppoſed in ſuch a progreſſion, the next " to which (though it exceed that finite number by an unit on- " ly) is ſuppoſed infinite." From theſe ſuppoſitions it is inferred, " that in ſuch a progreſſion continued to infinity there are " finite numbers whoſe ſquares become infinite;" though it ſeems very evident, that a finite number taken any finite number of times can never produce more than a finite number. We may perceive from theſe inſtances, that it is not by founding the higher geometry on the doctrine of infinites we can propoſe to avoid

avoid the apparent inconfiftencies that have been objected to it; and fince an excellent author, who has always diftinguifhed himfelf as a clear and acute writer, has had no better fuccefs in eftablifhing it on thefe principles, it is better for us to avoid them. Thefe fuppofitions however may be of ufe, when employed with caution, for abridging computations in the inveftigation of theorems, or even for proving them where a fcrupulous exactnefs is not required; and we would not be underftood to affirm, that the methods of indivifibles and infinitefimals, by which fo many unconteftted truths have been difcovered, are without a foundation. We acknowledge further, that there is fomething marvellous in the doctrine of infinites, that is apt to pleafe and tranfport us; and that the method of infinitefimals has been profecuted of late with an acutenefs and fubtlety not to be parallelled in any other fcience. But geometry is beft eftablifhed on clear and plain principles; and thefe fpeculations are ever obnoxious to fome difficulties. If the greateft accuracy has been always required in this fcience, in reafoning concerning finite quantities, we apprehend that Geometricians cannot be too fcrupulous in admitting or treating of infinites, of which our ideas are fo imperfect. Philofophy probably will always have its myfteries. But thefe are to be avoided in geometry: and we ought to guard againft abating from its ftrictnefs and evidence the rather, that an abfurd philofophy is the natural product of a vitiated geometry.

It is juft at the fame time to acknowledge, that they who firft carried geometry beyond its ancient limits, and they who have fince enlarged it, have done great fervice, by defcribing plainly the methods which they found fo advantageous for this purpofe, (though they might appear exceptionable in fome refpects,) that others might proceed with the fame facility to improve it. Some of them have been fo cautious as to verify their difcoveries by demonftrations in the ftricteft form; and others were able to have done this, had they not chofe rather to employ their time in extending the fcience. At firft, the variation from the ancient method was not fo confiderable, but that it was eafy to have recourfe to it, when it fhould be thought neceffary

for the satisfaction of such as required a scrupulous exactness. The Geometricians in the mean time made great improvements. They had the accurate method and examples of ARCHIMEDES before them, by which they might try their discoveries. These served to keep them from error, and the new methods facilitated their progress. Thus their views enlarged; and problems, that appeared at first sight of an insuperable difficulty, were afterwards resolved, and came at length to be despised as too simple and easy. The mensuration of parabolas, hyperbolas, spirals of all the higher orders, and of the famous cycloid, were amongst the earliest productions of this period; some of which seem to have been discovered by several Geometricians almost at the same time. It is not necessary for our purpose to describe more particularly what discoveries were made by TORRICELLI, Mess. de FERMAT and de ROBERVAL, GREGORY à Sto. Vincentio, &c. by whom the theorems of ARCHIMEDES were continued, and applied to the mensuration of various figures.

The *Arithmetica infinitorum* of Dr. WALLIS was the fullest treatise of this kind that appeared before the invention of the method of fluxions. ARCHIMEDES had considered the sums of the terms in an arithmetical progression, and of their squares only, (or rather the limits of these sums, described above,) these being sufficient for the mensuration of the figures he had examined. Dr. WALLIS treats this subject in a very general manner, and assigns like limits for the sums of any powers of the terms, whether the exponents be integers or fractions, positive or negative. Having discovered one general theorem that includes all of this kind, he then compounded new progressions from various aggregates of these terms, and enquired into the sums of the powers of these terms, by which he was enabled to measure accurately, or by approximation, the areas of figures without number. But he composed this treatise (as he tells us) before he had examined the writings of ARCHIMEDES, and he proposes his theorems and demonstrations in a less accurate form. He supposes the progressions to be continued to infinity, and investigates, by a kind of induction, the proportion of the sum of the powers to the product that would arise by taking the

greatest

greatest power as often as there are terms. His demonstrations, and some of his expressions (as when he speaks of quantities more than infinite) have been excepted against. But it was not very difficult to demonstrate the greatest part of his propositions in a stricter method; and this was effected afterwards by himself and others in various instances. He chose to describe plainly a method which he had found very commodious for discovering new theorems; and it must be owned, that this valuable treatise contributed to produce the great improvements which soon followed after. A like apology may be made for others who have promoted this doctrine since his time, but have not given us rigid demonstrations. In general, it must be owned, that if the late discoveries were deduced at length, in the very same method in which the ancients demonstrated their theorems, the life of man could hardly be sufficient for considering them all: so that a general and concise method, equivalent to theirs in accuracy and evidence, that comprehends innumerable theorems in a few general views, may well be esteemed a valuable invention.

Cavalerius was sensible of the difficulties, as well as the advantages that attended his method. He speaks as if he foresaw that it should be afterwards delivered in an unexceptionable form, that might satisfy the most scrupulous Geometrician; and leaves this *Gordian knot*, as he expresses himself, to some *Alexander*. Its form indeed was soon altered, and many improvements were made by the Mathematicians who prosecuted it since his time that deserve to be mentioned with esteem. But the method still remained liable to some exceptions, and was thought to be less perfect than that of the ancients on several grounds.

Sir Isaac Newton accomplished what Cavalerius wished for, by inventing the method of Fluxions, and proposing it in a way that admits of strict demonstration, which requires the supposition of no quantities but such as are finite, and easily conceived. The computations in this method are the same as in the method of infinitesimals; but it is founded on accurate principles, agreeable to the ancient geometry. In it, the premisses

and conclusions are equally accurate, no quantities are rejected as infinitely small, and no part of a curve is supposed to coincide with a right line. The excellency of this method has not been so fully described, or so generally attended to, as it seems to deserve; and it has been sometimes represented as on a level in all these respects with the method of infinitesimals. The chief design of the following treatise is, to shew its advantages in a clearer and fuller light, and to promote the design of the great inventor, by establishing the higher geometry on plain principles, perfectly consistent with each other and with those of the ancient Geometricians.

The method of demonstration which we make most use of in this treatise, was first suggested to us from a particular attention to Sir ISAAC NEWTON's brief reasoning in that place of his principles of philosophy where he first published the elements of this doctrine. After the greatest part of the following treatise was writ, we had the pleasure to observe, that Geometricians of the first rank had recourse to it long ago on several occasions, as a method of the strictest kind. Mr. de FERMAT, in a letter to GASSENDUS, and Mr. HUYGENS, in his *Horologium oscillatorium*, have employed it for completing the demonstrations of some theorems that were proposed by GALILEUS, and proved by him in a less accurate manner; and Dr. BARROW has demonstrated by it a theorem concerning the tangents of curve lines. The approbation which it appears to have had from so good judges, encouraged us to publish the following treatise; where it is applied for demonstrating the method of Fluxions. The chief pursuit of Geometricians for some time has been to improve their general methods. In proportion as these are valuable, it is important that they be established above all exception: and since they save us so much time and labour, we may allow the more for illustrating these methods themselves.

THE

THE

ELEMENTS

OF THE

Method of FLUXIONS,

Demonstrated after the Manner of the

Ancient Geometricians.

BOOK I.

Of the Fluxions of Geometrical Magnitudes.

CHAP. I.

Of the Grounds of this Method.

1. THE mathematical sciences treat of the relations of quantities to each other, and of all their affections that can be subjected to rule or measure. They treat of the properties of figures that depend on the position and form of the lines or planes that bound them, as well as those that depend on their magnitude; of the direction of motion, as well as its velocity; of the composition and resolution of quantities, and of every thing of this nature that is susceptible of a regular determination. We enquire

 into

into the relations of things, rather than their inward essences, in these sciences. Because we may have a clear conception of that which is the foundation of a relation, without having a perfect or adequate idea of the thing it is attributed to *, our ideas of relations are often clearer and more distinct than of the things to which they belong; and to this we may ascribe in some measure the peculiar evidence of the mathematics. It is not necessary that the objects of the speculative parts should be actually described, or exist without the mind; but it is essential, that their relations should be clearly conceived, and evidently deduced: and it is useful, that we should chiefly consider such as correspond with those of external objects, and may serve to promote our knowledge of nature.

2. In our pursuits after knowledge, we sometimes consider things as they appear to be in themselves; sometimes we judge of them from their causes, and sometimes by their effects. In ordinary enquiries, but especially in philosophy, we employ one or more of these methods according as we find ground for applying them. The two last may be no less satisfactory than the first, when there is a sufficient foundation for them; and by carrying our enquiries to the springs and principles of things, our knowledge of them becomes more perfect, and our views more extensive. In geometry, there are various ways of discovering the affections and relations of magnitudes that correspond to these general methods of enquiry. In the common geometry, we suppose the magnitudes to be already formed, and compare them or their parts, immediately, or by the intervention of others of the same kind, to which they have a relation that is already known. In the doctrine which we propose to explain and demonstrate in this treatise, we have recourse to the genesis of quantities, and either deduce their relations, by comparing the powers which are conceived to generate them; or, by comparing the quantities that are generated, we discover the relations of these powers and of any quantities that are supposed to be represented by them. The power by which magnitudes are conceived to be generated in geometry, is motion; and therefore we must begin with some account of it.

* Essay concerning the human understanding, book 2. chap. 25. § 8.

3. No

3. No quantities are more clearly conceived by us than the limited parts of ſpace and time. They conſiſt indeed always of parts; but of ſuch as are perfectly uniform and ſimilar. Thoſe of ſpace exiſt together; thoſe of time flow continually: but by motion they become the meaſures of each other reciprocally. The parts of ſpace are permanent; but being deſcribed ſucceſſively by motion, the ſpace may be conceived to flow as the time. The time is ever periſhing; but an image or repreſentation of it is preſerved and preſented to us at once in the ſpace deſcribed by the motion.

4. Time is conceived to flow always in an uniform courſe, that ſerves to meaſure the changes of all things. When the ſpace deſcribed by motion flows as the time, ſo that equal parts of ſpace are deſcribed in any equal parts of the time, the motion is uniform; and the velocity is meaſured by the ſpace that is deſcribed in any given time. As this ſpace may be conceived to be greater or leſs, and to be ſuſceptible of all degrees of aſſignable magnitude; ſo may the velocity of the motion by which we ſuppoſe the ſpace to be always deſcribed in a given time. The velocity of an uniform motion is the ſame at any term of the time during which it continues. But motion is ſuſceptible of the ſame variations with other quantities, and the velocity in other inſtances may increaſe or decreaſe while the time increaſes. In theſe caſes, however, the velocity at any term of the time is accurately meaſured by the ſpace that would be deſcribed in a given time, if the motion was to be continued uniformly from that term.

5. Any ſpace and time being given, a velocity is determined by which that ſpace may be deſcribed in that given time: And, converſely, a velocity being given, the ſpace which would be deſcribed by it in any given time is alſo determined. This being evident, it does not ſeem to be neceſſary, in pure geometry, to enquire further what is the nature of this power, affection or mode, which is called *Velocity*, and is commonly aſcribed to the body that is ſuppoſed to move. It ſeems to be ſufficient for our purpoſe, that while a body is ſuppoſed in motion, it muſt be conceived to have ſome velocity or other at any term of the time during which it moves, and that we can demonſtrate accu-

rately what are the measures of this velocity at any term, in the enquiries that belong to this doctrine, as will appear in the course of this treatise; especially since it is the business of geometry, as we have observed already, to enquire into the measures, rather than unfold the hidden essences of things.

6. But perhaps this explication will not be thought sufficient, and it will be required that we should propose a definition of velocity in form. The excellent Dr. BARROW defines it to be the power by which a certain space may be described in a certain time. Some perhaps may scruple to ascribe power to a body, figure or point in motion. But it is to be observed, that it is of no consequence, in pure geometry, to what the power may be most properly attributed. It is indeed generally allowed, that if a body was to be left to itself from any term of the time of its motion, and was to be affected by no external influence after that term, it would proceed for ever with an uniform motion, describing always a certain space in a given time: and this seems to be a sufficient foundation for ascribing, in common language, the velocity to the body that moves, as a power. It is well known, that what is an effect in one respect, may be considered as a power or cause in another; and we know no cause in common philosophy, but what is itself to be considered as an effect: but this does not hinder us from judging of effects from such causes. However, if any dislike this expression, they may suppose any mover or cause of the motion they please, to which they may ascribe the power, considering the velocity as the action of this power, or as the adequate effect and measure of its exertion, while it is supposed to produce the motion at every term of the time. We have observed already, that the principles of this method are analogous to the general doctrine of powers, or may be considered as a particular application of it. As a power which acts continually and uniformly is measured by the effect that is produced by it in a given time, so the velocity of an uniform motion is measured by the space that is described in a given time. If the action of the power vary, then its exertion at any term of the time is not measured by the effect that is actually produced after that term in a given time, but by the effect that would have been produced

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ced if its action had continued uniform from that term: and, in the same manner, the velocity of a variable motion at any given term of time is not to be measured by the space that is actually described after that term in a given time, but by the space that would have been described if the motion had continued uniformly from that term. If the action of a variable power, or the velocity of a variable motion, may not be measured in this manner, they must not be susceptible of any mensuration at all. It will appear afterwards, in the course of this treatise, that the other principles of this method correspond with the plain maxims of the general doctrine of powers that are employed by us on every occasion, and are to be reckoned amongst the most common and evident notions. There are two fundamental principles of this method. The first is, That when the quantities which are generated are always equal to each other, the generating motions must be always equal. The second is the converse of the first, That when the generating motions are always equal to each other, the quantities that are generated in the same time must be always equal. The first is the foundation of the direct method of fluxions; the second, of the inverse method. But it is obvious, that they may be considered as cases of these two general principles: When the effects produced by two powers are always equal to each other, then (supposing that no other power of any kind affects their operations) these powers must be supposed to act equally at any term of the time; and, conversely, When the actions of two powers are always equal to each other at any term of the time, then the effects produced by them in the same time must be always equal.

7. This method is so well founded, that its rules and operations may be delivered in a way consistent with any general principles that are not repugnant to the most evident notions; though it is impossible for us, in treating of it, to keep to expressions that may appear equally consistent with every scheme of metaphysics. It has been frequently considered in a manner agreeable to the principles of those who suppose quantities to consist of indivisible or infinitely small elements. We are to proceed upon more strict and rigid principles: but it will be hardly possible for

for us to avoid always ſuch expreſſions as may have been ſome time or other matter of diſpute amongſt philoſophers. Their controverſies concerning continued and diſcrete quantity have not been thought to weaken the evidence of the common geometry. Nor can their diſputes * concerning motion affect the certainty of this method; ſince we have occaſion in it for no more than the moſt obvious notions of ſpace, time, motion and velocity, that cannot be ſaid to yield in clearneſs and evidence to the principles of the common geometry.

8. When we ſuppoſe that a body has ſome velocity or other at any term of the time during which it moves, we do not therefore ſuppoſe that there can be any motion in a term, limit or moment of time, or in an indiviſible point of ſpace: and as we ſhall always meaſure this velocity by the ſpace that would be deſcribed by it continued uniformly for ſome given finite time, it ſurely will not be ſaid that we pretend to conceive motion or velocity without regard to ſpace and time.

9. But to proceed: When any quantity is propoſed, all others of the ſame kind may be conceived to be generated from it; ſuch as are greater than it, by ſuppoſing it to be increaſed; ſuch as are leſs, by ſuppoſing it to be diminiſhed. In the common arithmetic, integer numbers are conceived to be produced by adding a given quantity or unit to itſelf continually, and fractions are produced by ſuppoſing it to be divided into ſuch parts as by a like addition would generate the given quantity itſelf. But in geometry, that all degrees of magnitude may be produced, and in ſuch a way as may found a general method of deriving their affections from their geneſis, we conceive the quantities to be increaſed and diminiſhed, or to be wholly generated by motion, or by a continual flux analogous to it. The quantity that is thus generated, is ſaid to flow, and called *a Fluent*.

10. Lines are generated by the motion of points; ſurfaces, by the motion of lines; ſolids, by the motion of ſurfaces; angles, by the rotation of their ſides; the flux of time being ſuppoſed to be always uniform. The velocity with which a line flows,

* De natura motûs & recta definitione, de cauſis ac differentiis, complura ſubtiliter argutantur Phyſici; quatum ferè Mathematicis nihil cordi vel curæ: ſufficere poteſt his quæ communis ſenſus agnoſcit. *Barrow, lect. geom.* 1.

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is the ſame as that of the point which is ſuppoſed to deſcribe or generate it. The velocity with which a ſurface flows, is the ſame as the velocity of a given right line, that, by moving parallel to itſelf, is ſuppoſed to generate a rectangle which is always equal to the ſurface. The velocity with which a ſolid flows, is the ſame as the velocity of a given plain ſurface, that, by moving parallel to itſelf, is ſuppoſed to generate an erect priſm or cylinder that is always equal to the ſolid. The velocity with which an angle flows, is meaſured by the velocity of a point, that is ſuppoſed to deſcribe the arch of a given circle, which always ſubtends the angle, and meaſures it. In general, all quantities of the ſame kind (when we conſider their magnitude only, and abſtract from their poſition, figure, and other affections) may be repreſented by right lines, that are ſuppoſed to be always in the ſame proportion to each other as theſe quantities. They are repreſented by right lines in this manner in the Elements, in the general doctrine of proportion, and by right lines and figures in the *Data* of EUCLID *. In this method likewiſe, quantities of the ſame kind may be repreſented by right lines, and the velocities of the motions by which they are ſuppoſed to be generated, by the velocities of points moving in right lines. All the velocities we have mentioned are meaſured, at any term of the time of the motion, by the ſpaces which would be deſcribed in a given time, by theſe points, lines or ſurfaces, with their motions continued uniformly from that term.

11. The velocity with which a quantity flows, at any term of the time while it is ſuppoſed to be generated, is called its *Fluxion* which is therefore always meaſured by the increment or decrement that would be generated in a given time by this motion, if it was continued uniformly from that term without any acceleration or retardation : or it may be meaſured by the quantity that is generated in a given time by an uniform motion which is equal to the generating motion at that term.

12. Time is repreſented by a right line that flows uniformly, or is deſcribed by an uniform motion; and a moment or termination of time is repreſented by a point or termination of

* See the preface to the *Data* by Marinus, near the end.

that line. A given velocity is reprefented by a given line, the fame which would be defcribed by it in a given time. A velocity that is accelerated or retarded, is reprefented by a line that increafes or decreafes in the fame proportion. The time of any motion being reprefented by the bafe of a figure, and any part of the time by the correfponding part of the bafe; if the ordinate at any point of the bafe be equal to the fpace that would be defcribed, in a given time, by the velocity at the correfponding term of the time continued uniformly, then any velocity will be reprefented by the correfponding ordinate. The fluxions of quantities are reprefented by the increments or decrements defcribed in the laft article which meafure them; and, inftead of the proportion of the fluxions themfelves, we may always fubftitute the proportion of their meafures.

13. When a motion is uniform, the fpaces that are defcribed by it in any equal times are always equal. When a motion is perpetually accelerated, the fpaces defcribed by it in any equal times that fucceed after one another, perpetually increafe. When a motion is perpetually retarded, the fpaces that are defcribed by it in any equal times that fucceed after one another, perpetually decreafe.

14. It is manifeft, converfely, that if the fpaces defcribed in any equal times are always equal, then the motion is uniform. If the fpaces defcribed in any equal times that fucceed after one another perpetually increafe, the motion is perpetually accelerated: For it is plain, that if the motion was uniform for any time, the fpaces defcribed in any equal parts of this time would be equal; and if it was retarded for any time, the fpaces defcribed in equal parts of this time that fucceed after one another would decreafe: both of which are againft the fuppofition. In like manner it is evident, that a motion is perpetually retarded, when the fpaces that are defcribed in any equal times that fucceed after one another perpetually decreafe. The following Axioms are as evident as that a greater or lefs fpace is defcribed in a given time, according as the velocity of the motion is greater or lefs.

AXIOM I.

15. *The ſpace deſcribed by an accelerated motion is greater than the ſpace which would have been deſcribed in the ſame time, if the motion had not been accelerated, but had continued uniform from the beginning of the time.*

AXIOM II.

The ſpace deſcribed by a motion while it is accelerated, is leſs than the ſpace which is deſcribed in an equal time by the motion that is acquired by that acceleration continued uniformly.

AXIOM III.

The ſpace deſcribed by a retarded motion is leſs than the ſpace which would have been deſcribed in the ſame time, if the motion had not been retarded, but had continued uniform from the beginning of the time.

AXIOM IV.

The ſpace deſcribed by a motion while it is retarded, is greater than the ſpace which is deſcribed in an equal time by the motion that remains after that retardation, continued uniformly.

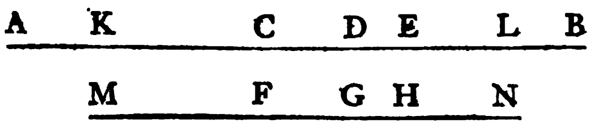
16. Before we proceed to enquire into the fluxions of quantities, it is neceſſary to premiſe the following general Theorems, which contain the grounds of this method. The two firſt are from the treatiſe of ARCHIMEDES concerning ſpiral lines.

THEOREM I.

The ſpaces deſcribed by an uniform motion are in the ſame proportion to each other as the times in which they are deſcribed.

Suppoſe a point to deſcribe the right line AB with an uniform motion. Let it deſcribe the ſpace CD in the time FG, and the ſpace DE in the time GH; then ſhall CD be to DE as FG is to GH.

Let KD and MG be any equimultiples of the lines CD and FG; and let DL and GN be any equimultiples of DE and GH. Becauſe the motion of the point in the line AB is ſuppoſed to be uniform, and



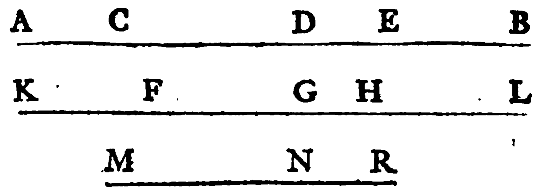
it deſcribes CD in the time FG, it will deſcribe any line equal to CD in a time equal to FG, and it will deſcribe KD, that is a multiple of CD, in the time MG, that is an equimultiple of FG. For the ſame reaſon, it will deſcribe DL in the time GN. But, when a motion is uniform, a greater ſpace is always deſcribed in a greater time: ſo that, if the ſpace KD exceed DL, the time MG will exceed GN; if KD be equal to DL, the time MG will be equal to GN; and if KD be leſs than DL, the time MG muſt be leſs than GN. Therefore (by def. 5. lib. 5. Elem.) CD is to DE as FG is to GH; that is, the ſpaces deſcribed by an uniform motion are in the ſame proportion to each other as the times in which they are deſcribed.

THEOREM II.

17. *The ſpaces deſcribed by an uniform motion are to each other in the ſame proportion as the ſpaces deſcribed in the ſame times by any other uniform motion.*

Suppoſe

Suppose two points to describe the right lines AB, KL with any uniform motions. Let CD and FG be described by them in the time MN; and let DE and GH be described by them in the time NR. Then, by the first theorem, the space CD is to the space DE as the time MN is to the time NR; and FG is to GH in the same ratio of MN to NR. Therefore CD is to DE as FG is to GH; that is, the spaces described by the first motion are in the same proportion to each other, as the spaces described in the same, or in equal times, by the second motion. The spiral of ARCHIMEDES being described by the composition of two uniform motions, one of which is rectilineal, the other circular, he had occasion, in demonstrating its properties, to make use of no more of the doctrine of motion than these two theorems. But, in establishing a general method for discovering the properties of curvilineal figures, we have occasion also for these that follow.



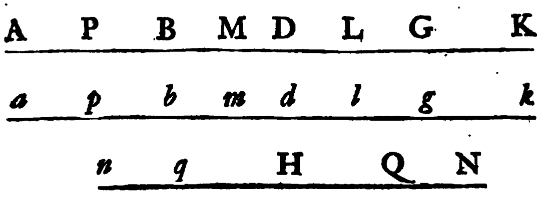
THEOREM III.

18. *If the spaces that are described in the same time by two motions, uniform or variable, be always equal to each other, the velocities of these motions must be equal at any term of the time.*

Suppose the points P and *p* to describe the lines AK, *ak* in the same time, with motions uniform or varied at pleasure, but so that the space described by P be always equal to the space described by *p* in the same time. Then shall the velocity of *p* at any term or moment of time be equal to the velocity of P at the same term or moment. This theorem is so evident, that it may seem to need no proof. If AK and *ak* be right lines, and we suppose *ak* to be placed upon AK, the point *p* will be always

always over the point P, and their velocities at the ſame term of the time cannot but be ſuppoſed equal, whether the motions be uniform or variable. But as this is a fundamental theorem in this doctrine, and holds whether the points P and *p* move in right lines or in curves, we ſhall demonſtrate its various caſes from the preceeding axioms.

19. *Caſe* 1. Suppoſe the motion of P to be uniform. Then, ſince equal ſpaces are deſcribed by P in any equal times, it follows, from the ſuppoſition, that equal ſpaces are alſo deſcribed by *p* in any equal times, and that its motion is alſo uniform. But the velocities



of uniform motions are equal when equal ſpaces are deſcribed by them in the ſame time; and therefore the velocities of P and *p* are in this caſe always equal to each other.

20. *Caſe* 2. Suppoſe that the motion of P is perpetually accelerated, or that the ſpaces deſcribed by it in any equal parts of time that ſucceed after one another, perpetually increaſe. Then the ſpaces deſcribed by *p* being always equal (by the ſuppoſition) to the ſpaces deſcribed in the ſame times by P, the ſpaces deſcribed by *p* in any equal parts of time that ſucceed each other muſt alſo perpetually increaſe, and the motion of *p* muſt be perpetually accelerated, (by art. 14.) Let P come to D, and *p* to *d*, at the ſame term or moment of time; and their velocities at that term ſhall be equal. For, if they are not equal, ſuppoſe firſt the velocity of *p* to exceed the velocity of P; let DG and *dg* be any equal ſpaces deſcribed after that term by the points P and *p* in the time repreſented by HN. Then, becauſe the motions of theſe points are perpetually accelerated during the time HN, it follows from the firſt axiom, that the ſpace which would be deſcribed in that time by *p* with its motion at *d* continued uniformly, is leſs than *dg*, which is deſcribed by it with an accelerated motion in the ſame time; and it follows from the ſecond axiom, that the ſpace which would be deſcri-

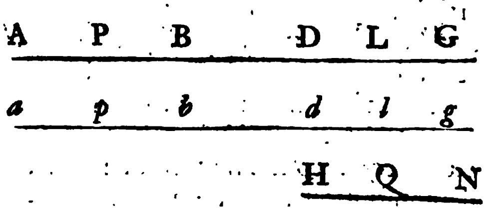
described by the point P with the motion it has acquired at G continued uniformly for the time HN, is greater than DG, which is described by it in the same time before that motion is acquired. Therefore, since *dg* is equal to DG, (by the supposition,) the velocity of *p* at *d* is less than the velocity of P at G. But the velocity of *p* at *d* is supposed to be greater than the velocity of P at D; and therefore it may be supposed equal to the velocity of P at some intermediate term of the time HN, as when it comes to a point L betwixt D and G. Let *dl* be equal to DL, and let HQ be the time in which P and *p* describe the equal spaces DL and *dl* with their accelerated motions. Then, by the first axiom, the space which would be described in the time HQ by the motion of *p* at *d* continued uniformly, is less than *dl*, which is described in the same time by its accelerated motion; and, by the second axiom, the space which would be described in the same time HQ by the point P with the motion it has acquired at L continued uniformly, is greater than DL, which was described by it in the same time before that motion was acquired. But, by the supposition, *dl* is equal to DL; and therefore a less space would be described in the same time by the motion of *p* at *d* continued uniformly, than by the motion of P at L continued uniformly: so that the velocity of *p* at *d* must be less than the velocity of P at L. But these velocities were supposed equal; and these being contradictory, it appears that the velocity of *p* at *d* is not greater than the velocity of P at D. In the same manner it is shewn, that the velocity of P at D is not greater than the velocity of *p* at *d*. Therefore the velocities of P and *p* are equal at this or any other term of the time of their motion.

21. The velocities with which the points P and *p* come to D and *d* may be shewn to be equal, from the same principles, without supposing their motions to be continued after the term H, by considering the spaces described by them before that term. Let BD and *bd* be any equal spaces described by the points P and *p* with their accelerated motions in the time *n*H, before they come to D and *d*; and if their velocities be not then equal, let the velocity of *p* at *d* exceed the velocity of P at D. By the second axiom, the space that would be described

bed by the motion of P at D continued uniformly in the time *n*H is greater than BD. By the first axiom, the space that would be described in the same time *n*H by the motion of *p* at *b* continued uniformly, is less than *bd* or BD. Therefore the velocity of P at D is greater than the velocity of *p* at *b*: but it is supposed to be less than the velocity of *p* at *d*; and consequently, it may be supposed equal to the velocity of *p* at some intermediate term *q* of the time *n*H, when *p* comes to some point *m* betwixt *b* and *d*. Let MD be equal to *md*, and the spaces MD, *md* will be described in the same time *q*H by the points P and *p*. By the second axiom, the space which would be described in the time *q*H by the motion of P at D continued uniformly, is greater than MD. By the first axiom, the space which would be described in the same time *q*H by the motion of *p* at *m* continued uniformly, is less than *md* or MD. Therefore the velocity of P at D is greater than the velocity of *p* at *m*: but they were supposed equal, and these are contradictory. It appears therefore, that the velocity of *p* at *d* is not greater than the velocity of P at D; and in the same manner it is shewn, that the velocity of P at D is not greater than the velocity of *p* at *d*: and therefore these velocities must be equal to each other.

22. *Case* 3. Suppose that the motion of P is perpetually retarded, or that the spaces perpetually decrease which are described by it in any equal times that succeed each other. Then, by the supposition, the spaces described by *p* in any equal succeeding times also decrease, and its motion is also perpetually retarded. If the velocity of P at D be not equal to the velocity of *p* at *d*, let it first be greater. Suppose DG and *dg* to be equal spaces described, in any time HN, by the points P and *p* with their motions continued after the term H, at which they are supposed to come to D and *d*. Because their motions are perpetually retarded, it follows from the third axiom, that the space which would be described, in the time HN, by *p* with its motion at *d* continued uniformly, is greater than *dg*, which is described in the same time by *p* with a retarded motion; and it follows from the fourth axiom, that the space which would be described by P with the motion that remains at G continued uni-

uniformly for a time equal to HN, is leſs than DG, (or *dg*,) which is deſcribed in the time HN before the motion of P is reduced to the velocity which it retains at the term G. Therefore the velocity of *p* at *d* is greater than the velocity of P at G: and ſince it is ſuppoſed leſs than the velocity of P at D, it may therefore be ſuppoſed equal to ſome intermediate velocity of P, as to that with which it comes to ſome point L betwixt D and G. Let *dl* be equal to DL; and let theſe ſpaces be deſcribed by the points P and *p* with their retarded motions in the time HQ. Then, by the third axiom, the ſpace that would be deſcribed in the time HQ by the motion of *p* at *d* continued uniformly is greater than *dl*; and, by the fourth axiom, the ſpace that would be deſcribed in the ſame time HQ by the motion of P at L continued uniformly is leſs than DL or *dl*. Therefore the velocity of *p* at *d* is greater than the velocity of P at L: but they were ſuppoſed equal; and theſe are contradictory. It appears, therefore, that the velocity of P at D is not greater than the velocity of *p* at *d*; and ſince it is ſhewn, in the ſame manner, that the velocity of *p* at *d* is not greater than the velocity of P at D, theſe velocities muſt therefore be equal. It is eaſy to ſhew that theſe velocities are equal, by conſidering the ſpaces BD and *bd* that are deſcribed by P and *p* before they come to D and *d*, without ſuppoſing their motions to be continued after that term.



23. If the motions of the points P and *p* are ſometimes accelerated and ſometimes retarded, then DG and *dg*, or BD and *bd*, muſt be ſuppoſed to be ſpaces deſcribed by them, either while they are accelerated only, or while they are retarded only. We have chiefly in view, in theſe theorems, ſuch motions as are uniform, and ſuch as are increaſed or diminiſhed by a continued acceleration or retardation. But the demonſtration may be extended to theſe caſes alſo, where the motions are ſuppoſed to be increaſed or diminiſhed at certain terms of the time by aſſignable aug-

ments or decrements at once, by ſuppoſing BD and DG to be the ſpaces deſcribed in the interval of time betwixt two ſuch ſucceeding terms, or in a part of that interval. Or, in the 20th article, inſtead of ſuppoſing, in the latter part of the demonſtration, that the velocity of *p* at *d* is equal to the velocity of P at L, ſome intermediate place betwixt D and G, we may ſuppoſe that it is either equal or greater than the velocity of P at L. In the laſt article, we may ſuppoſe the motion of *p* at *d* to be either equal to the motion of P at L, or leſs than it; and thus the demonſtration may be accommodated to thoſe caſes, when the motions of P and *p* are perpetually increaſing or decreaſing, but not in a continued manner. See below, art. 44, 45 and 46.

THEOREM IV.

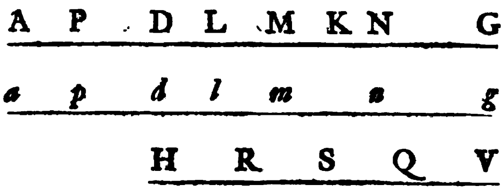
24. *If the velocities of two motions are always equal to each other, the ſpaces deſcribed by them in the ſame time are always equal.*

Suppoſe the points P and *p* to deſcribe the right lines AG and *ag* with any motions, uniform or variable, but ſo as that the velocity of *p* at any term or moment of time be always equal to the velocity of P at the ſame term. Let DG and *dg* be any ſpaces deſcribed by them in the ſame time HV; and DG ſhall be always equal to *dg*.

Caſe 1. If the motions of the points P and *p* are uniform, it is evident, that ſince theſe motions are equal, (by the ſuppoſition,) the ſpaces deſcribed by them in the ſame time muſt be equal; and therefore in this caſe DG is equal to *dg*.

25. *Caſe* 2. Suppoſe the motions of the points P and *p* to be perpetually accelerated while they deſcribe the ſpaces DG and *dg*; and if *dg* be not equal to DG, let it firſt be equal to any line DK leſs than DG. Let the right line HV, which repreſents the time, be divided by a continual biſection into equal parts HR, RS, SQ, QV, till the time repreſented by one of theſe parts, as QV, be leſs than the time in which the point P deſcribes KG. Let DL, LM, MN, NG be the ſpaces deſcribed by P, and *dl*, *lm*, *mn*, *ng* the ſpaces deſcribed by *p*, in the equal

equal parts of the time represented by HR, RS, SQ, QV; and, by the supposition, the velocities of P at the points L, M and N will be respectively equal to the velocities of *p* at the points *l*, *m* and *n*. The space *ng*, which is described by *p* in the time QV with an accelerated motion, is greater than the space which

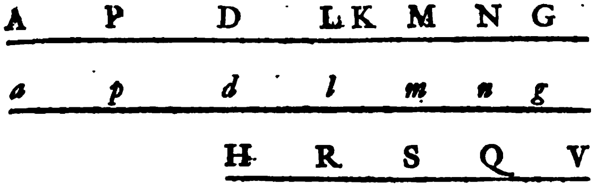


would be described in the same time by the motion of *p* at *n* continued uniformly, by the first axiom. The velocity of *p* at *n* is equal to the velocity of P at N; and, by the second axiom, the space which would be described in the time QV by the point P with its motion at N continued uniformly, is greater than MN, which was described by it in an equal time before its velocity at N was acquired. Therefore *ng* is greater than MN. In the same manner it appears, that *mn* is greater than LM, and *lm* greater than DL; so that *lg* is greater that DN, and *dg* is surely greater than DN. But DN is greater than DK; for the time QV is supposed to be less than the time in which KG is described by the point P; so that NG being described in the time QV, it must be less than KG, and DN must be greater than DK. Therefore *dg* being greater than DN, it must be greater than DK: but it was supposed equal to DK; and these are contradictory. It appears, therefore, that the space *dg* is not less than DG. In the same manner it is shewn, that DG is not less than *dg*; and therefore these spaces DG and *dg*, which are described by the points P and *p* in the same time, are equal to each other.

26. *Case* 3. Suppose the motions of the points P and *p* to be perpetually retarded; and if *dg* be not equal to DG, let it first be equal to a line KG less than DG. Let the time HV be divided by a continual bisection into the equal parts HR, RS, SQ, QV, till the time HR be less than that in which P describes DK. Let DL, LM, MN, NG be the spaces described by P, and *dl*, *lm*, *mn*, *ng* be the spaces described by *p*, in the equal times HR, RS, SQ, QV. Then DL shall be less than DK, because it is described by the

 point

point P in a leſs time; and the velocities of P at the points L, M, N and G will be reſpectively equal to the velocities of *p* at the points *l*, *m*, *n* and *g*, by the ſuppoſition. The motions of the points P and *p* being perpetually retarded, it follows, from



the fourth axiom, that *mn* is greater than the ſpace which would be deſcribed in the time SQ (or QV) by the motion of *p* at *n* continued uniformly. But the velocity of *p* at *n* is equal to the velocity of P at N; and, by the third axiom, the ſpace which would be deſcribed in the time QV, by the motion of P at N continued uniformly, is greater than NG, which is deſcribed by P with a retarded motion in the ſame time. Therefore *mn* is greater than NG. In the ſame manner it appears, that *lm* is greater than MN, *dl* greater than LM; and, conſequently, that *dn* is greater than LG: ſo that *dg* is ſurely greater than LG, and therefore greater than KG, which is leſs than LG. But *dg* was ſuppoſed equal to KG; and theſe being contradictory, it follows, that *dg* is not leſs than DG. In the ſame manner it appears, that DG is not leſs than *dg*; and therefore the ſpaces DG and *dg* are equal. By joining theſe caſes together, the theorem is demonſtrated, when the motions of the points P and *p* are ſometimes accelerated, and ſometimes retarded.

THEOREM V.

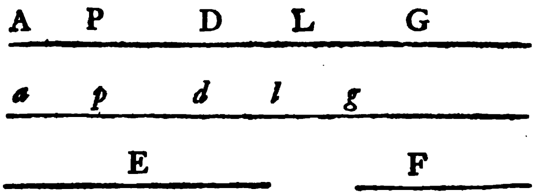
27. *When the ſpaces deſcribed by two motions in the ſame time are always in an invariable ratio to each other, the velocites of theſe motions are always in the ſame invariable ratio.*

Let DG and *dg* be any two ſpaces deſcribed by the points P and *p* in the ſame time; and let DG be always to *dg* as E is to

to F. Then the velocity of P at any term of the time ſhall be to the velocity of p at the ſame term in the ſame invariable ratio of E to F. In the firſt place, if the motion of P be uniform, ſo that equal ſpaces be deſcribed by it in any equal times, equal ſpaces will be alſo deſcribed by p in any equal times, and its motion will be alſo uniform. But the velocities of uniform motions are in the ſame proportion as the ſpaces deſcribed by them in the ſame time; and therefore, in this caſe, the velocity of P is to the velocity of p as DG is to dg, or as E is to F.

28. *Caſe* 2. If the motion of the point P be perpetually accelerated, then (becauſe the ſpaces deſcribed by P and p in the ſame time are always to each other in a given invariable ratio) the ſpaces deſcribed by p in equal times perpetually increaſe, and its motion is alſo perpetually accelerated. In the ſame time that the points P and p deſcribe the ſpaces DG and dg with their accelerated motions, the point P with its motion at G continued uniformly would deſcribe a greater ſpace than DG, (by the ſecond axiom;) and the point p with its motion at d continued uniformly would deſcribe a leſs ſpace than dg, by the firſt axiom. Therefore the velocity of the point P at G is to the velocity of p at d in a greater ratio than DG is to dg, or E is to F. From which it follows, that if the velocity of P at D was to the velocity of p at d in a leſs ratio than that of E to F, then the velocity of P at ſome intermediate term betwixt D and G might be to the velocity of p at d in the very ſame ratio as E is to F. But this is impoſſible; for ſuppoſing L to be ſuch a term, let the ſpace DL be to dl as E is to F, and theſe ſpaces will be deſcribed by P and p in the ſame time, by the ſuppoſition. It follows, from the ſecond axiom, that the point P with its motion at L continued uniformly would deſcribe a greater ſpace than DL, in the ſame time that it deſcribes the ſpace DL with its accelerated motion; and, by the firſt axiom, the point p with its motion at d continued uniformly would deſcribe a leſs ſpace than dl in the ſame time. Therefore the velocity of P at L is to the velocity of p at d in a greater ratio than DL is to dl, or E to F; ſo that the velocity of P at L might be to the velocity of p at d in the ſame ratio as E is to F, and in a greater ratio at the ſame time; which is abſurd. It appears, there-

therefore, that the ratio of the velocity of P at D to the velocity of *p* at *d* cannot be less than that of E to F. Suppose now that ratio to be greater than the ratio of E to F. In the same time that P and *p* describe DG and *dg* with their accelerated motions, the point P with its motion at



D continued uniformly would describe a less space than DG, (by the first axiom,) and the point *p* with its motion at *g* continued uniformly would describe a greater space than *dg*, by the second axiom: so that the velocity of P at D is to the velocity of *p* at *g* in a less ratio than DG is to *dg*, or E is to F. Therefore, if the velocity of P at D was to the velocity of *p* at *d* in a greater ratio than that of E to F, the velocity of P at D might be to the velocity of *p* at some intermediate term betwixt *d* and *g* in the same ratio as E is to F. But this is impossible. For supposing *l* to be such a term, and DL to be to *dl* in the same ratio as E is to F, these spaces would be described by P and *p* in the same time, by the supposition; and a less space than DL would be described in that time by the motion of P at D continued uniformly, (by the first axiom,) but a greater space than *dl* would be described in the same time by the motion of *p* at *l* continued uniformly, by the second axiom; so that the velocity of P at D would be to the velocity of *p* at *l* in a less ratio than that of DL to *dl*, or of E to F. It appears, therefore, that the velocity of P at D is to the velocity of *p* at *d* neither in a greater nor less ratio than that of E to F, but precisely in this ratio.

29. *Case* 3. If the motion of P is perpetually retarded, the motion of *p* must also be perpetually retarded, because the spaces described by these points in equal times decrease in the same proportion. In this case, a less space than DG would be described by the motion of P at G continued uniformly, (by the fourth axiom,) and a greater space than *dg* would be described by the motion of *p* at *d* continued uniformly, (by the third

third axiom,) in the time that P and *p* with their retarded motions deſcribe DG and *dg*; ſo that the velocity of P at G is to the velocity of *p* at *d* in a leſs ratio than that of DG to *dg*, or of E to F. Therefore, if we ſuppoſe that the velocity of P at D is to the velocity of *p* at *d* in a greater ratio than that of E to F, it follows, that the velocity of P at ſome point betwixt D and G, as L, might be to the velocity of *p* at *d* in the ſame ratio as E is to F. But this is impoſſible. For, ſuppoſing that DL is to *dl* as E is to F, theſe ſpaces will be deſcribed by P and *p* in the ſame time; and a leſs ſpace than DL would be deſcribed in that time by the motion of P at L continued uniformly, (by the fourth axiom;) but a greater ſpace than *dl* would be deſcribed in the ſame time by the motion of *p* at *d* continued uniformly, by the third axiom; ſo that the velocity of P at L is to the velocity of *p* at *d* in a leſs ratio than that of DL to *dl*, or of E to F. It is impoſſible, therefore, that the velocity of P at D can be to the velocity of *p* at *d* in a greater ratio than that of E to F. Nor can that ratio be leſs than the ratio of E to F. For in the ſame time that P and *p* with their retarded motions deſcribe DG and *dg*, a greater ſpace than DG would be deſcribed by the motion of P at D continued uniformly, (by the third axiom,) and a leſs ſpace than *dg* would be deſcribed by the motion of *p* at *g* continued uniformly, by the fourth axiom: ſo that the velocity of P at D is to the velocity of *p* at *g* in a greater ratio than DG is to *dg*, or E is to F. Therefore, if the velocity of P at D was to the velocity of *p* at *d* in a leſs ratio than that of E to F, it might be to the velocity of *p* at ſome point as *l* betwixt *d* and *g* in the ſame ratio as E is to F. But this is impoſſible. For, ſuppoſing that DL is to *dl* as E is to F, the ſpaces DL and *dl* will be deſcribed by P and *p* with their retarded motions in the ſame time; and a greater ſpace than DL would be deſcribed in that time by the motion of P at D continued uniformly, (by the third axiom,) but a leſs ſpace than *dl* would be deſcribed in the ſame time by the motion of *p* at *l* continued uniformly, by the fourth axiom: ſo that the velocity of P at D is to the velocity of *p* at *l* in a greater ratio than that of DL to *dl*, or of E to F. It appears, therefore, that the velocity of P at D is to the velocity

city of p at d in a ratio that is neither greater nor leſs than that of E to F, but is preciſely the ſame ratio.

30. This theorem may be alſo demonſtrated, by conſidering the ſpaces that are deſcribed by the points P and p before they come to D and d, whether their motions be ſuppoſed to be continued after that term or not. By joining theſe caſes together, the demonſtration becomes general; and the ſame obſervation we made in the 23d article is to be applied here.

LEMMA I.

31. *If* A *be to* B *in a greater ratio than* E *is to* F, *and* C *be to* D *in a greater ratio than* E *is to* F; *then ſhall the ſum of the antecedents* A *and* C *be to the ſum of the conſequents* B *and* D *in a greater ratio than that of* E *to* F.

For let G be to B as E is to F, and H to D as E to F; then the ſum of G and H ſhall be to the ſum of B and D as E is to F. But A is greater than G, becauſe A is to B in a greater ratio than G is to B; and C is greater than H, becauſe C is to D in a greater ratio than H is to D. Therefore the ſum of A and C is greater than the ſum of G and H; and, conſequently, the ſum of A and C is to the ſum of B and D in a greater ratio than E is to F. In the ſame manner it appears in general, that if there be any number of ratios, each greater than the ratio of E to F, the ſum of all the antecedents ſhall be to the ſum of all the conſequents in a greater ratio than that of E to F.

32. It appears, in the ſame manner, that if A be to B in a leſs ratio than E is to F, and C be alſo to D in a leſs ratio than E is to F; then the ſum of A and C ſhall be to the ſum of B and D in a leſs ratio than that of E to F: and, in general, if there be any number of ratios, each leſs than that of E to F, then the ſum of all the antecedents ſhall be to the ſum of all the conſequents in a leſs ratio than that of E to F.

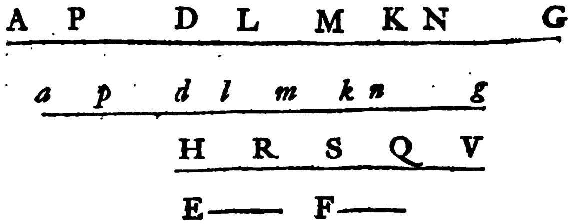
THE-

THEOREM VI.

33. ***When the velocities of any motions are always to each other in an invariable ratio, the ſpaces deſcribed by them in the ſame time are always in the ſame ratio.***

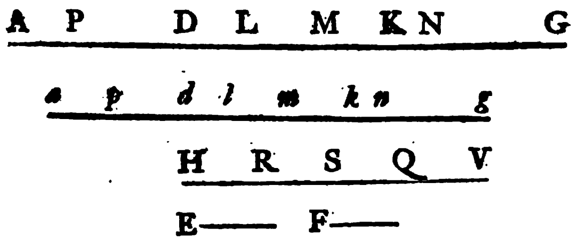
Suppoſe that the velocity of the point P is always to the velocity of *p* in the invariable ratio of E to F; let DG and *dg* be any ſpaces deſcribed by theſe points in the ſame time HV. Then ſhall DG be to *dg* as E is to F. This is the converſe of the preceeding theorem, and may be demonſtrated by it and the fourth theorem: but it may be alſo demonſtrated immediately from the axioms in the following manner. In the firſt place, if the motions of P and *p* are uniform, it is evident, that the ſpaces deſcribed by them in the ſame time are in the ſame proportion as the velocities of the motions; and therefore, in this caſe, DG is to *dg* as E is to F.

34. *Caſe* 2. Suppoſe the motions of P and *p* to be perpetually accelerated; and, if the ratio of DG to *dg* be greater than that of E to F, let DK be to *dg* as E is to F; and DK ſhall be leſs than DG. Let the time HV be divided by a continual biſection into the equal parts HR, RS,



SQ, QV, till the time QV be leſs than that in which P deſcribes KG. Let DL, LM, MN, NG be the ſpaces deſcribed by P, and *dl*, *lm*, *mn*, *ng* be the ſpaces deſcribed by *p*, in theſe equal times HR, RS, SQ, QV. Then ſhall NG be leſs than KG, becauſe it is deſcribed by P in a leſs time than KG; and the velocities of P at L, M, N and G, will be to the velocities of *p* at *l*, *m*, *n* and *g* reſpectively, as E is to F, by the ſuppoſition. It follows, from the ſecond axiom, that MN is leſs than the ſpace which would be deſcribed, in the time QV, by the motion of P at N continued uni-

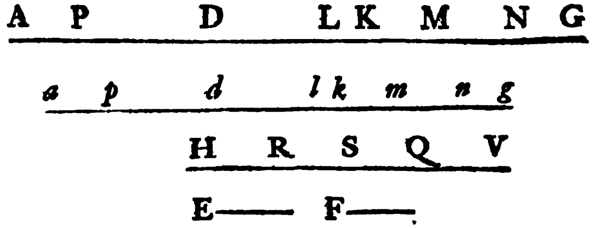
uniformly; and, by the first axiom, ng is greater than the space which would be described in the same time by the motion of p at n continued uniformly. Therefore MN is to ng in a less ratio than the velocity of P at N is to the velocity of p at n; that is, in a less ratio than E is to F. In the same manner, LM is to mn, DL to lm, and, consequently, (by art. 32.) DN to lg in a less ratio than that of E to F. Therefore DN is surely to dg in a less ratio than E is to F, or DK to dg; and, consequently, DN is less than DK. But DN is greater than DK, because NG is less than KG; and these are contradictory. Therefore DG is to dg in a ratio that is not greater than that of E to F. If it be said, that DG is to dg in a less ratio than that of E to F; let DG be to dk as E is to F, and dk will be less than dg. Suppose the time HV to be divided into the equal parts HR, RS, SQ, QV, till QV be less than the time in which p describes kg. Let DL, LM, MN, NG be the spaces described



by P, and dl, lm, mn, ng the spaces described by p, in the equal times HR, RS, SQ, QV. Then, since NG is greater than the space which would be described, in the time QV, by the motion of P at N continued uniformly, (by the first axiom;) and mn is less than the space which would be described, in the same time, by the motion of p at n continued uniformly, (by the second axiom:) it follows, that NG is to mn in a greater ratio, than that of the velocity of P at N to the velocity of p at n, which (by the supposition) is the ratio of E to F. In the same manner, MN is to lm, LM to dl, and therefore LG to dn (by art. 31.) in a greater ratio than that of E to F. Therefore DG is surely in a greater ratio to dn, than that of E to F, or that of DG to dk; and, consequently, dn is less than dk. But the time QV, in which p describes ng, was supposed to be less than the time in which p describes kg: therefore ng is less than kg, and dn is greater than dk: and these being contradictory, it fol-

follows, that the ratio of DG to *dg* is not leſs than that of E to F. It appears, therefore, that DG is to *dg* as E is to F.

35. *Caſe* 3. Suppoſe the motions of P and *p* to be perpetually retarded; and, if DG be to *dg* in a greater ratio than that of E to F, let KG be to *dg* as E is to F, and KG will be leſs than DG. Let the time HV be divided into equal parts, HR, RS, SQ, QV, till HR become leſs than the time in which P deſcribes DK; let DL, LM, MN, NG be deſcribed by the point P, and *dl*, *lm*, *mn*, *ng* be deſcribed by the point *p*, in theſe equal times. Then, becauſe DL is deſcribed by P in a leſs time than DK, DL is leſs than DK, and LG is greater than KG. But LG muſt be ſuppoſed to be leſs than KG: For it eaſily appears, from the third and fourth axioms, that NG is to *mn* in a leſs ratio than that of the velocity of P at N to the velocity of *p* at *n*, or that of E to F; and, in the ſame manner, it appears, that MN is to *lm*, LM to *dl*, and conſequently (by art. 32.) LG to *dn*, in a leſs ratio than E is to F: ſo that LG is ſurely to *dg* in a leſs ratio than E to F, or KG to *dg*: From which it follows, that LG is leſs than KG. If the ratio of DG to *dg* be ſaid to be leſs than that of E to F, let DG be to *kg* (leſs than *dg*) as E is to F. Suppoſe, as before, the time HV to be ſubdivided into



the equal parts HR, RS, SQ, QV, till HR become leſs than the time in which *p* deſcribes *dk*; and the ſpaces DG and *dg* being ſubdivided as formerly, *dl* ſhall be leſs than *dk*, and *lg* greater than *kg*. But *lg* muſt be leſs than *kg*. For it appears, from the fourth axiom, that DL is greater than the ſpace which would be deſcribed, in the time HR, by the motion of P at L continued uniformly; and *lm* is leſs than the ſpace which would be deſcribed, in the ſame time, by the motion of *p* at *l* continued uniformly, by the third axiom: ſo that DL is to *lm* in a greater ratio, than that of the velocity of P at L to the veloci-

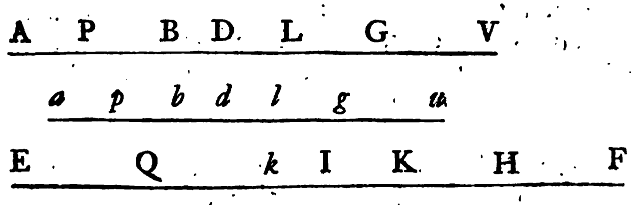
ty of p at l, or that of E to F. In the same manner, LM is to mn, MN is to ng, and consequently (by art. 31.) DN to lg, in a greater ratio than E is to F. Therefore DG is to lg in a greater ratio than E is to F, or DG to kg; and, consequently, lg is less than kg. Thus it appears, that when the velocities of the points P and p are always to each other in any invariable ratio, the spaces described by them in the same time are always in the same ratio.

THEOREM VII.

36. *When the space described by a motion is always equal to the sum of the spaces described in the same time by any other motions, the velocity of the first motion is always equal to the sum of the velocities of the other motions.*

Let the three points P, p and Q move in the lines AV, au and EF, and describe the spaces DG, dg and IH in the same time; and let IH be always equal to the sum of DG and dg: then shall the velocity of Q be always equal to the sum of the velocities of P and p. If the motions of the points P and p are both uniform, then equal spaces being described by them in any equal times, the spaces described by Q in equal times will be also equal, and its motion will be uniform; and the velocity of Q will be to the sum of the velocities of P and p as IK is to the sum of DG and dg, and therefore in a ratio of equality.

37. If the motion of P be continually accelerated, and the motion of p be either uniform or accelerated, the motion of Q is also continually accelerated, (by art. 14.) and if the sum of the velocity of P at D,



added to the velocity of p at d, was greater than the velocity of Q at I, this sum might be supposed equal to the velocity of

Q

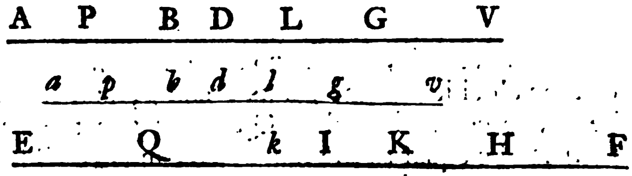
Q at ſome ſubſequent term, as when it comes to K. Let P and *p* deſcribe DL and *dl* in the ſame time that Q deſcribes IK; and it follows, from the laſt article, that the point Q, with its motion at K continued uniformly, would in this time deſcribe a ſpace equal to the ſum of the ſpaces that would be deſcribed, in the ſame time, by the points P and *p*, with their motions at D and *d* continued uniformly, that is, a ſpace leſs than the ſum of DL and *dl*, (by the firſt axiom;) and therefore leſs than IK. But the point Q, with its motion at K continued uniformly, would deſcribe a greater ſpace than IK in that time, by the ſecond axiom; and theſe being contradictory, it follows, that the velocity of Q at I is not leſs than the ſum of the velocities of P at D and *p* at *d*. If the velocity of Q at I was greater than the ſum of theſe velocities, then its velocity at ſome preceeding term, as when it arrived at *k*, might be equal to that ſum; and, ſuppoſing BD, *bd* and *k*I to be deſcribed by P, *p* and Q in the ſame time, the point Q, with its motion at *k* continued uniformly, might deſcribe a ſpace greater than the ſum of BD and *bd* (or *k*I) in the time it deſcribes *k*I with its accelerated motion, againſt the firſt axiom. This theorem is demonſtrated in the ſame manner from the third and fourth axioms, when the motion of P is perpetually retarded, and the motion of *p* is either uniform or retarded.

38. When the motion of P is continually accelerated, but the motion of *p* retarded, the motion of Q may be accelerated, uniform, or retarded. Suppoſe, firſt, that the motion of Q is accelerated; and if the velocity of Q at I was greater than the velocity of P at D added to the velocity of *p* at *d*, it might be ſuppoſed equal to the velocity of *p* at *d* added to the velocity of P at ſome ſubſequent term, as when it comes to L. But this is impoſſible. For, ſuppoſing that this could be admitted, and that *dl* and IK are deſcribed by *p* and Q in the time P deſcribes DL, the point P, with its motion at L continued uniformly, would deſcribe a greater ſpace than DL in that time, (by the ſecond axiom;) and the point *p*, with its motion at *d* continued uniformly, would deſcribe a greater ſpace than *dl* in the ſame time, (by the third axiom:) ſo that the point Q, with its motion at I continued uniformly, would de-

ſcribe a greater ſpace than the ſum of DL and *dl* (or IK) in that time. But, becauſe the motion of Q is ſuppoſed to be continually accelerated, it follows, from the firſt axiom, that the point Q, with its motion at I continued uniformly, would deſcribe a leſs ſpace than IK in that time; and theſe being contradictory, it appears, that the velocity of Q at I is not equal to the velocity of P at L added to the velocity of *p* at *d*, and cannot exceed the ſum of the velocity of P at D added to the velocity of *p* at *d*. If the velocity of Q at I was leſs than this ſum, it might be equal to the velocity of *p* at *d* added to the velocity of P at ſome preceeding term of the time, as when it came to B. But this is impoſſible. For, if it could be admitted, then, ſuppoſing *bd* and *kI* to be deſcribed by *p* and Q while P deſcribes BD, the point P, with its motion at B continued uniformly, would deſcribe a leſs ſpace than BD (by the firſt axiom) in that time; and the point *p*, with its motion at *d* continued uniformly, would deſcribe a leſs ſpace than *bd* (by the fourth axiom) in the ſame time: ſo that the point Q, with its motion at I continued uniformly, would deſcribe in this time a ſpace leſs than the ſum of BD and *bd*, or *kI*. But the motion of Q being continually accelerated, the point Q, with its motion at I continued uniformly, would deſcribe a greater ſpace than *kI* in that time, by the ſecond axiom: and theſe being contradictory, it is evident, that the velocity of Q at I is not equal to the velocity of P at B added to the velocity of *p* at *d*, and cannot be leſs than the velocity of P at D added to the velocity of *p* at *d*. If the motion of Q be uniform, while the motion of P is accelerated and the motion of *p* is retarded, it is demonſtrated, in the ſame manner, that the velocity of Q at I is equal to the velocity of P at D added to the velocity of *p* at *d*.

39. If the velocity of Q be continually retarded, (the reſt remaining as in the laſt article,) and its velocity at I was greater than the velocity of P at D added to the velocity of *p* at *d*, it might be equal to the velocity of *p* at ſome preceeding term of the time, as when it came to *b*, added to the velocity of P at D. But this is impoſſible. For, ſuppoſing that this is admitted, and that P and Q deſcribe BD and *kI* in the time *p* deſcribes *bd*, the point *p*, with its motion at *b* continued uniformly,

ly, would deſcribe in that time a ſpace greater than bd, (by the third axiom;) the point P, with its motion at D continued uniformly, would deſcribe a greater ſpace than BD in the ſame time, (by the ſecond axiom :) and therefore the point Q, with its motion at I continued uniformly, would deſcribe in this time a ſpace greater than the ſum of BD and bd, or kI. But the motion of Q being continually retarded, it would deſcribe a leſs ſpace than kI with its motion at I continued uniformly in that time, by the fourth axiom : and theſe are contradictory.



If the velocity of Q at I was leſs than the velocity of P at D added to the velocity of p at d, it might be ſuppoſed equal to the velocity of P at D added to the velocity of p at ſome ſubſequent term of the time, as when it comes to l; and (by the third and fourth axioms) it might deſcribe, with its motion at I continued uniformly, a leſs ſpace than the ſum of DL and dl, or IK, in the ſame time that by its retarded motion it deſcribes IK; but it would deſcribe, with the ſame uniform motion, in the ſame time, a greater ſpace than IK, by the third axiom : and theſe are contradictory. Therefore the velocity of Q at I is equal to the velocity of P at D added to the velocity of p at d.

40. If the ſpace deſcribed by Q be always equal to the ſum of the ſpaces deſcribed in the ſame time by three or more points, it is eaſy, from what has been ſhewn, to extend the demonſtration to theſe caſes, by ſubſtituting always one point in place of two : And it appears, in general, that the velocity of Q at any term of the time, is equal to the ſum of the velocities of all the other points at the ſame term.

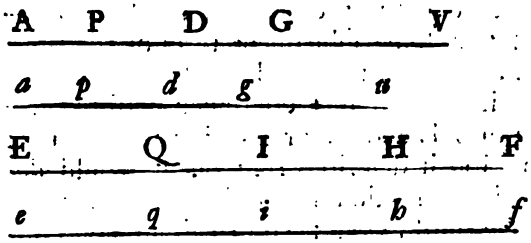
41. It follows, from what has been demonſtrated, that when the ſpace deſcribed by any point p is always equal to the difference of the ſpaces deſcribed in the ſame time by the points Q and P, then the velocity of p is always equal to the difference of the velocities of theſe points.

THE-

THEOREM VIII.

42. *When the velocity of a motion is always equal to the sum of the velocities of any other motions, the space described by it is always equal to the sum of the spaces described in the same time by these other motions.*

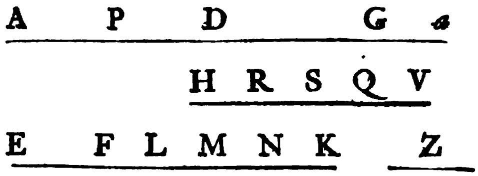
Let the velocity of the point Q be always equal to the sum of the velocities of the points P and *p*; and let IH, DG and *dg* be any spaces described by Q, P and *p* in the same time: then shall IH be equal to the sum of DG and *dg*. This theorem may be demonstrated immediately from the axioms, in the same manner as the fourth and sixth; but more briefly thus: Suppose the point *q* to describe, upon the line *ef*, a space *ib* always equal to the sum of DG and *dg*, then (by the seventh theorem) the velocity of *q* must be always equal to the sum of the velocities of P and *p*, and therefore is equal to the velocity of Q. From which it follows, by the fourth theorem, that IH is always equal to *ib*; and therefore IH is equal to the sum of DG and *dg*.



43. It follows, from this theorem, that when the velocity of any point *p* is always equal to the difference of the velocities of two other points Q and P, the space *dg* described by *p* is always equal to the difference betwixt IH and DG, which are described in the same time by the points Q and P.

44. In demonstrating these theorems, we have supposed that every motion is either uniform, continually accelerated, or continually retarded; or that the time may be distinguished into parts, during each of which the motion is reducible to one or other

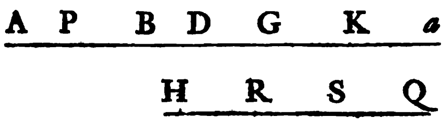
other of thoſe kinds. For though the velocity of a motion may, at certain terms of the time, be increaſed or diminiſhed at once by a given or aſſignable quantity; it is impoſſible that a motion can be increaſed or diminiſhed in this manner perpetually, or at every term of the time. If ſuch a motion could be ſuppoſed, its velocity would exceed all conceivable velocities in the leaſt time that could be aſſigned. This, if it ſeem to need a proof, may be demonſtrated in the following manner. Let the point P deſcribe any ſpace DG, upon the line A*a*, in the time HV; and let its velocity at G be to its velocity at D in any aſſignable ratio, as in that of EK to EF. The velocity of P may be ſuppoſed, at certain terms of the time,



to be increaſed at once, by a quantity that may be in the ſame ratio to the velocity of P at D, as a given line Z is to EF; but it cannot be ſuppoſed to be increaſed in every aſſignable part of the time by this quantity, how minute ſoever it may be. For, ſuppoſing that this could be admitted, let FK be divided by a continual biſection into equal parts, FL, LM, MN, NK, till each of theſe parts become leſs than Z; and let HV, which repreſents the time, be divided into the ſame number of equal parts, HR, RS, SQ, QV. Then, by what we have ſuppoſed, the velocity of P muſt be increaſed, in the time HR, by a quantity that is to the velocity of P at D as Z is to EF; and therefore, ſince Z is greater than FL, the velocity of P at the end of the time HR, is to its velocity at the beginning of that time in a greater ratio than EL is to EF. In like manner, the velocity of P being increaſed in each of the ſubſequent times RS, SQ, QV, by the quantity repreſented by Z, which is greater than LM, MN or NK; it follows, that the velocity of P at the end of the time HV, (that is, when it is ſuppoſed to come to G,) is to the velocity it had at the beginning of that time, when it was at D, in a greater ratio than EK is to EF: but the ratio of theſe velocities was ſuppoſed to be the ſame as that of EK to EF; and theſe are contradictory. It appears, in the

ſame manner, that the velocity of P cannot be ſuppoſed to decreaſe, ſo as to be diminiſhed by a given quantity how minute ſoever, repreſented by Z, at every term of the time, or in every aſſignable part of it.

45. When a motion is accelerated or retarded in a continued manner, it is evident, that, from any given term, a part of the time may be taken ſo ſmall, that the difference of the velocities at the beginning and end of this time, may be leſs than the difference of any given unequal velocities; and that the ratio of thoſe velocities approaches to a ratio of equality, by diminiſhing the time, ſo that it may come nearer to that ratio than any given ratio of inequality. It follows from this, that if BD, DG and GK be deſcribed by ſuch a motion in the equal times HR, RS, SQ; then, by diminiſhing the time HQ, the ratio of GK to BD will approach nearer to a ratio of equality than any ratio of inequality that can be aſſigned. For, if the motion be accelerated, BD muſt be greater than the ſpace which would be deſcribed in the time HR by the motion at B continued uniformly, (by the firſt axiom;) and GK is leſs than the ſpace which would be deſcribed, in the ſame time, by the motion at K continued uniformly, (by the ſecond axiom:) ſo that GK is greater than BD, but in a leſs ratio than the velocity at K is greater than the velocity at B; and therefore in a ratio, that, by diminiſhing the time HQ, may approach nearer to a ratio of equality, than any aſſignable ratio of inequality. If the motion of P be retarded in a continued manner, it appears, from the third and fourth axioms, that the ratio of BD to GK is leſs than the ratio of the velocity at B to the velocity at K; ſo that, by diminiſhing the time HQ, the ratio of BD to GK may approach nearer to a ratio of equality, than any given ratio of inequality.



46. It follows, converſely, that if the ratio of GK to BD approach to a ratio of equality by diminiſhing the time HQ, ſo as to come nearer to it than any given ratio of inequality, and this obtain whatever part of the time be repreſented by HQ; then the motion muſt be increaſed or diminiſhed in a con-

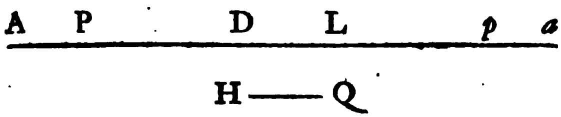
continued manner. For, if the motion perpetually increaſe, then the velocity at G is to the velocity at D in a leſs ratio, than GK is to BD, (by the firſt and ſecond axioms;) and therefore in a ratio that (by the ſuppoſition) may come nearer to a ratio of equality, than any given ratio of inequality: ſo that the motion muſt be accelerated in a continued manner. If the motion perpetually decreaſe, then the velocity at D is to the velocity at G in a leſs ratio than BD is to GK, (by the third and fourth axioms;) and therefore in a ratio that may approach to a ratio of equality nearer than any given ratio of inequality: ſo that the motion is retarded in a continued manner. The motions that are increaſed or diminiſhed in a continued manner, are thus diſtinguiſhed from thoſe that at certain terms are increaſed by a given quantity, but betwixt thoſe terms are either uniform, or are accelerated or retarded in a continued manner. It appears eaſily, from what has been ſhewn, that, in the third and fifth theorems, if the motion of P be accelerated or retarded in a continued manner, the motion of *p* is alſo accelerated or retarded in a continued manner; and that, in the ſeventh theorem, when the motions of P and *p* are accelerated or retarded in a continued manner, the motion of Q is accelerated or retarded in the ſame manner.

THEOREM IX.

47. *When a point* P *deſcribes a line* A*a with a motion of any kind, and another point p deſcribes the ſame ſpaces on this line* A*a, in equal times, but in a contrary order, and with an oppoſite direction; the velocities of theſe points at any given term of the line are equal.*

Suppoſe that the point P proceeding from A towards *a* deſcribes any ſpace DL in any time HQ; and that the point *p*, in moving from *a* towards A, deſcribes always that ſpace DL in the ſame time HQ in which it was deſcribed by P: Then the velocity of *p* at D muſt be equal to the velocity of P at D.

When the motion of the point P is uniform, it follows, from the ſuppoſition, that the motion of p is alſo uniform, and that their velocities are always equal. Let the motion of P be continually accelerated; and it follows, from the ſuppoſition, that the motion of p is continually retarded. If the velocity of the



point p at D was greater than the velocity of P at D, it might be ſuppoſed equal to the velocity of P at ſome ſubſequent term of the time, as when it comes to L. But this is impoſſible: For the point P, with its motion at L continued uniformly, would deſcribe a greater ſpace than DL, in the time HQ, (by the ſecond axiom;) and the point p, with its motion at D continued uniformly, would deſcribe a leſs ſpace than DL, in that time, (by the fourth axiom:) ſo that the velocity of p at D is leſs than the velocity of P at L. If the velocity of P at D was greater than the velocity of p at D, it might be ſuppoſed equal to the velocity of p at ſome term before p came to D, as to its velocity at L. But this alſo is impoſſible: For the point P, with its motion at D continued uniformly, would deſcribe a leſs ſpace than DL in the time HQ, (by the firſt axiom;) and the point p, with its motion at L continued uniformly, would deſcribe a greater ſpace than LD in the ſame time, (by the third axiom:) ſo that the velocity of p at L is greater than the velocity of P at D. It appears, therefore, that the velocity of p at any term of the line Aa, as D, is neither greater nor leſs than the velocity of P at the ſame term, but equal to it. If the motion of P is continually retarded, the motion of p muſt be continually accelerated; and the demonſtration is the ſame as in the former caſe.

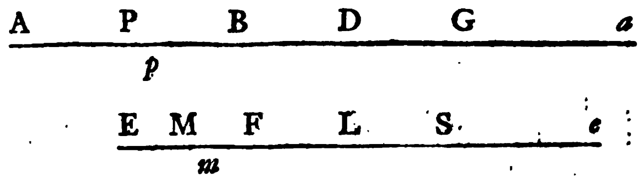
48. In the following articles, we ſuppoſe, that while the points P and p deſcribe the line Aa, the points M and m deſcribe the line Ee; and that EM is determined in any regular manner from AP, ſo that when AP is equal to the abſciſs of any figure, EM is always equal to the correſponding ordinate. Then, if Em be determined in the ſame manner from Ap, it is evident, that when Ap becomes equal to AP, Em becomes equal to

to EM; and that, according as AP is greater or leſs than A*p*, EM is greater or leſs than E*m*. If P and *p* come together to any point D, then M and *m* ſhall come at the ſame time to ſome point L upon the line E*e*; and, according as DP is greater or leſs than D*p*, LM is greater or leſs than L*m*. It is alſo evident, that the motion of P being uniform, if the motion of M be accelerated or retarded in a continued manner; then, the motion of *p* being alſo uniform, the motion of *m* muſt be accelerated or retarded in a continued manner. If this ſeem to need any proof, it may be eaſily deduced from the 45th and 46th articles.

THEOREM X.

49. *The motions of the points* P *and p in the line* A*a* *being uniform, let* EM *be always determined in any regular manner from* AP, *and* E*m* *be determined in the ſame manner from* A*p*; *then the velocity of the point* M, *at any term of the line* E*e*, *ſhall be to the velocity of the point m, at the ſame term of that line, as the velocity of* P *is to the velocity of p.*

Let the points P and *p* come together to D; let M and *m* come at the ſame time to L: and the velocity of M at L ſhall be to the velocity of *m* at L, as the velocity of the uniform motion of P is to the velocity of the uniform motion of *p*. If the motion of M is uniform, the motion of *m* is alſo uniform. For, if the point *m* deſcribe the ſpaces FL, LS in any equal times, the point *p*

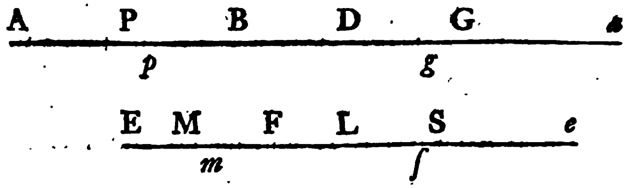


will deſcribe equal ſpaces BD and DG in theſe equal times; and, the motion of P being uniform, it will deſcribe the ſame ſpaces in equal times. But while P deſcribes BD and DG, M deſcribes FL and LS with its uniform motion; and therefore FL is

is equal to LS: and the ſpaces being equal which are deſcribed by *m* in any equal times, its motion muſt be uniform. The velocity of M is to the velocity of P as FL is to BD; and the velocity of *m* is to the velocity of *p* in the ſame ratio. Therefore the velocity of M is to the velocity of *m* as the velocity of P is to the velocity of *p*.

50. If the motion of M is continually accelerated, the motion of *m* is alſo accelerated continually. In this caſe, if the velocity of *m* at L was to the velocity of *p* in a greater ratio, than the velocity of M at L is to the velocity of P; then the velocity of *m* at L might be to the velocity of *p*, as the velocity of M at ſome ſubſequent term of the line E*e* is to the velocity of P. Suppoſe the point S to be ſuch a term, and let P deſcribe DG while M deſcribes LS. Then, becauſe the motion of the point M is continually accelerated, it would deſcribe a greater ſpace than LS with its motion at S continued uniformly, in the time P deſcribes DG, (by the ſecond axiom.) From which it follows, that the velocity of M at S is to the velocity of P in a greater ratio than LS is to DG; and therefore the velocity of *m* at L is to the velocity of *p* in a ratio that is alſo greater than that of LS to DG. But, in the ſame time that *p* deſcribes DG with an uniform motion, the point *m* deſcribes LS with an accelerated motion; and it would deſcribe a leſs ſpace than LS with its motion at L continued uniformly, in this time, (by the firſt axiom.) Therefore the velocity of *m* at L is to the velocity of *p* in a leſs ratio than LS is to DG: and theſe being contradictory, it appears, that the ratio of the velocity of *m* at L to the velocity of *p*, is not greater than the ratio of the velocity of M at L to the velocity of P. If it was a leſs ratio, then the velocity of *m* at ſome ſubſequent term of the line E*e* might be to the velocity of *p*, as the velocity of M at L is to the velocity of P. Suppoſe the point *ſ* to be that term; let *p* deſcribe D*g* while *m* deſcribes L*ſ*: and ſince the point *m*, with its motion at *ſ* continued uniformly, would deſcribe a greater ſpace than L*ſ* while *p* deſcribes D*g* with its uniform motion, (by the ſecond axiom;) it follows, that the velocity of *m* at *ſ* is to the velocity of *p* in a greater ratio than L*ſ* is to D*g*: and therefore the velocity of M at L is to the velocity

velocity of P in a ratio that is also greater than that of L*s* to D*g*. But while P describes D*g* with an uniform motion, the point M describes L*s* with an accelerated motion; and it would describe a less space than L*s* with its motion at L continued uniformly, in this time, (by the first axiom.) Therefore the velocity of M at L is to the velocity of P in a less ratio than L*s* is to D*g*. And these being contradictory, it appears, that the velocity of *m* at L is to the velocity of *p* neither in a greater nor in a less ratio, than the velocity of M at L is to the velocity of P. Therefore the velocity of M at L is to the velocity of *m* at L, as the velocity of P is to the velocity of *p*.

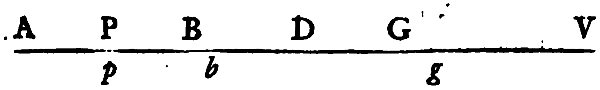


In like manner, this theorem is demonstrated from the third and fourth axioms, when the motion of M is continually retarded. In either case, it may be demonstrated from the same principles, by considering the spaces described by the points M and *m* before they come to L, whether their motions be continued after that term or not. An example of the manner how this is done, was given in the 21st article.

51 It is demonstrated, in the same manner, that EM being always determined from AP in any regular manner, and E*m* being determined from A*p* in the same manner; if the motions of P and M be uniform, and M come to L when P comes to D, then the velocity of *m* at L shall be to the constant velocity of M, as the velocity of *p* at D is to the constant velocity of P. If the motion of *p* is accelerated continually, the motion of *m* is also accelerated continually: and if the velocity of *m* at L was to the velocity of M in a greater ratio, than the velocity of *p* at D is to the velocity of P, it might be supposed that the velocity of *m* at L is to the velocity of M, as the velocity of *p* at *g* is to the velocity of P; that is, (supposing DG, D*g*, LS, L*s* to be described by the points P, *p*, M, *m* respectively, in the same time,) in a greater ratio than D*g* is to DG, (by the second axiom,) or (because the motions of P and

and M are uniform, and D*g* is to DG as L*s* is to LS) in a greater ratio than L*s* is to LS. But the velocity of *m* at L is to the velocity of M in a less ratio than L*s* is to LS, (by the first axiom;) and these being contradictory, it appears, that the ratio of the velocity of *m* at L to the velocity of M is not greater, than the ratio of the velocity of *p* at D to the velocity of P. In the same manner, it is shewn, that it is not a less ratio; and therefore the velocity of *m* at L is to the velocity of M, as the velocity of *p* at D is to the velocity of P. When the motions of *p* and *m* are continually retarded, the demonstration is deduced, in like manner, from the third and fourth axioms.

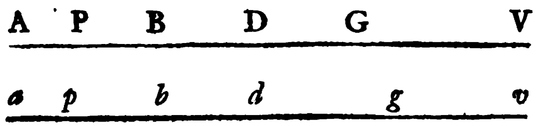
52. The motion of the point P in the line AV being uniform, but the motion of *p* variable, let their velocities at D be equal; let the time in which *p* describes *b*D be equal to the time in which P describes BD; and let D*g* and DG be also described by them in any equal times. Then, if the motion of *p* be perpetually accelerated, *b*D shall be always less than BD, (by



the second axiom,) because *b*D is described by *p* with an accelerated motion, and BD is described in an equal time by P with an uniform motion equal to that which *p* acquires at D. But D*g* is greater than DG, (by the first axiom,) because D*g* is described by *p* with an accelerated motion, and DG is described in an equal time by an uniform motion equal to the motion of *p* at D. If the motion of *p* be perpetually retarded, then (by the fourth axiom) *b*D is greater than BD, and (by the third axiom) D*g* is less than DG.

53. Let the motion of P in the line AV be uniform; and the motion of *p* in the line *av* be accelerated or retarded continually, (that is, let its velocity increase or decrease from one degree to another, by passing through all the intermediate degrees,) while P describes any spaces BD and DG; let *p* describe the spaces *bd* and *dg*; and, the motion of *p* being first accelerated, let *bd* be always less than BD, but *dg* greater than DG: then the velocity of *p* at *d* shall be equal to the constant velocity of P. For a greater space than *dg* would be described by the

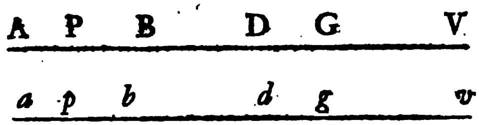
the point *p*, with its motion at *g* continued uniformly, in the ſame time P deſcribes DG, (by the ſecond axiom;) and therefore the velocity of P is leſs than the velocity of *p* at *g*. A leſs ſpace than *bd* would be deſcribed by the point *p*, with its motion at *b* continued uniformly, in the ſame time P deſcribes BD, (by the firſt axiom:) and therefore the velocity of P is greater than the velocity of *p* at *b*. The motion of *p* is ſuppoſed to be accelerated in a continued manner; and therefore, the conſtant velocity of P being greater than the velocity of *p* at *b*, but leſs than the velocity of *p* at *g*, it muſt be equal to the velocity of *p* at ſome intermediate term of the ſpace *bg*. But it is demonſtrated, in the ſame manner, that the velocity of P is greater than the velocity of *p* at any term before *d*, and leſs than the velocity of *p* at any term after *d*. Therefore the velocity of P is equal to the velocity of *p* at *d*. If the motion of *p* be retarded continually, and *bd* be always greater than BD, but *dg* leſs than DG; it is demonſtrated, in the ſame manner, from the third and fourth axioms, that the conſtant velocity of P is equal to the velocity of *p* at *d*.



54. Let the motion of the point P be alſo accelerated; but, the motion of *p* being more accelerated, let *bd* be always leſs than BD, but *dg* greater than DG. In this caſe, the velocity of P at D is leſs than the velocity of *p* at *g*, becauſe a leſs ſpace than DG would be deſcribed by the former continued uniformly, and a greater ſpace than *dg* would be deſcribed by the latter continued uniformly, in the ſame time, (by the firſt and ſecond axioms.) The velocity of P at D is greater than the velocity of *p* at *b*, becauſe a greater ſpace than BD would be deſcribed by the former continued uniformly, and a leſs ſpace than *bd* (which is itſelf leſs than BD) would be deſcribed by the latter continued uniformly, in the ſame time, (by the ſame axioms.) Therefore, the motion of the point *p* being accelerated continually, the velocity of P at D muſt be equal to the velocity of *p* at ſome intermediate term of the ſpace *bg*. But, in the ſame manner as we have ſhewn, that it is greater than the velocity

of *p* at *b*, and leſs than the velocity of *p* at *g*, it is demonſtrated to be greater than the velocity of *p* at any term before *p* comes to *d*, and to be leſs than the velocity of *p* at any term after it paſſes *d*. Therefore the velocity of P at D is equal to the velocity of *p* at *d*.

55. Let the motion of the point *p* be now retarded continually, and the motion of P be alſo retarded; let *bd* be always greater than BD, and *dg* leſs than DG, (which happens when the motion of *p* is more retarded than the motion of P.) The velocity of P at D is greater than the velocity of *p* at *g*, becauſe a greater ſpace than DG would be deſcribed by the former continued uniformly, and a leſs ſpace than *dg* would be deſcribed by the latter continued uniformly, in the ſame time, (by the third and fourth axioms.) The velocity of P at D is leſs than the velocity of *p* at *b*, (by the ſame axioms.) Therefore, ſince the motion of *p* is ſuppoſed to decreaſe in a continued manner, the velocity of P at D muſt be equal to the velocity of



p at ſome intermediate term of the ſpace *bg*. But, in the ſame manner as we have ſhewn, that the velocity of P at D is greater than the velocity of *p* at *g*, but leſs than the velocity of *p* at *b*, it is demonſtrated that the velocity of P at D is greater, or leſs, than the velocity of *p* at any term of the ſpace *bg*, *d* only excepted. Therefore the velocity of P at D is equal to the velocity of *p* at *d*.

56. When the motion of *p* is retarded continually, and the motion of P is accelerated continually; and *bd* is always greater than BD, but *dg* leſs than DG: then the velocity of *p* at *b* is greater than the velocity of P at B, becauſe a greater ſpace than *bd* would be deſcribed by the former continued uniformly, and a leſs ſpace than BD would be deſcribed by the latter continued uniformly, in the ſame time, (by the firſt and third axioms.) The velocity of *p* at *g* is leſs than the velocity of P at G, becauſe a leſs ſpace than *dg* would be deſcribed by the former continued uniformly, and a greater ſpace than DG would be deſcribed by the latter continued uniformly, in the ſame time, (by

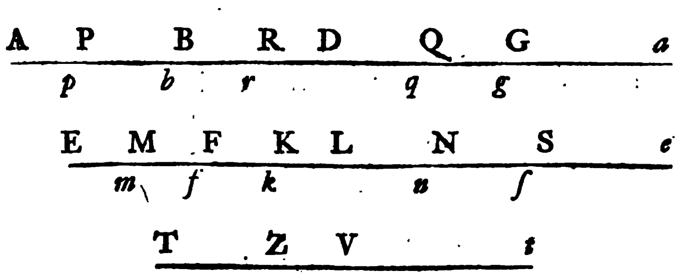
(by the ſecond and fourth axioms.) Therefore, ſince the motions of P and p increaſe and decreaſe in a continued manner, their velocities muſt be equal at ſome intermediate term of the time in which they deſcribe BG and bg. But, in the ſame manner as we have ſhewn that their velocities are unequal at B and b, and at G and g, it is demonſtrated that their velocities are unequal at any term of the time, that only when they come to D and d excepted. Therefore, in this caſe alſo, the velocity of P at D is equal to the velocity of p at d.

THEOREM XI.

57. *The motion of the point* P *upon the line* A*a being uniform, and the motion of the point p upon the ſame line being accelerated or retarded continually, let their velocities be equal at* D. *Then,* EM *being always determined from* AP *in any regular manner, and* E*m being determined from* A*p in the ſame manner; when* P *comes to* D, *let* M *come to* L *with a motion that is accelerated or retarded continually; and the velocity of m at* L *ſhall be equal to the velocity of* M *at* L.

In the firſt place, let the motion of M be continually accelerated, and the motion of p continually retarded. Let the points P, p, M and m deſcribe the ſpaces BD, bD, FL and fL reſpectively, in the time TV; and the ſpaces DG, Dg, LS, L$ſ$ in the time Vt. Then (by art. 52.) bD is greater than BD, and Dg leſs than DG. From which it follows, (art. 48.) that fL is greater than FL, and L$ſ$ leſs than LS. In this caſe, the motion of m may be uniform, accelerated, or retarded. Let it firſt be uniform. By the firſt axiom, the point M, with its motion at F continued uniformly, would deſcribe in the time TV a ſpace leſs than FL, which is itſelf leſs than fL, the ſpace deſcribed in the ſame time by the point m; ſo that the velocity of M at F is leſs than the conſtant velocity of m. Therefore, ſince the motion of M is accelerated in a continued man-

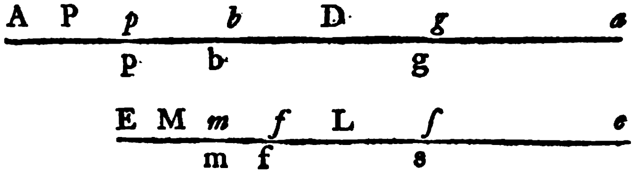
ner, by the ſuppoſition; if its motion at L be greater than the velocity of *m*, its velocity at ſome intermediate term of the ſpace FL, as K, muſt be equal to the conſtant velocity of *m*.



Let P, *p*, M and *m* deſcribe the ſpaces RD, *r*D, KL and *k*L in the ſame time ZV; and becauſe *r*D exceeds RD, (art. 52.) *k*L muſt be greater than KL. The point M, with its motion at K continued uniformly, would deſcribe in the time ZV a ſpace leſs than KL, (by the firſt axiom,) which is leſs than *k*L, that is deſcribed in the ſame time by the point *m* with its uniform motion. Therefore the velocity of M at K is leſs than the conſtant velocity of *m*. But they were equal; and theſe being contradictory, it appears, that the velocity of M at L is not greater than the conſtant velocity of *m*. The velocity of *m* is leſs than the velocity of M at S, becauſe L*ſ* is deſcribed by the former in the time V*t*, and a greater ſpace than LS (which exceeds L*ſ*) would be deſcribed in the ſame time by the latter continued uniformly, by the ſecond axiom. Therefore, if the velocity of *m* be greater than the velocity of M at L, it muſt be equal to the velocity of M at ſome intermediate term of the ſpace LS, as N. While M deſcribes LN, let P, *p* and *m* deſcribe DQ, D*q* and L*n*; and D*q* being leſs than DQ, (by art. 52.) L*n* is therefore leſs than LN. The point M, with its motion at N continued uniformly, would deſcribe a greater ſpace than LN (by the firſt axiom) while *m* with its uniform motion deſcribes L*n*; and therefore the velocity of *m* is leſs than the velocity of M at N. But they were equal; and theſe are contradictory. Therefore the velocity of *m* is neither greater nor leſs than the velocity of M at L, but preciſely equal to it. The demonſtration proceeds by the ſame ſteps when the velocity of M is accelerated, that of *p* retarded,

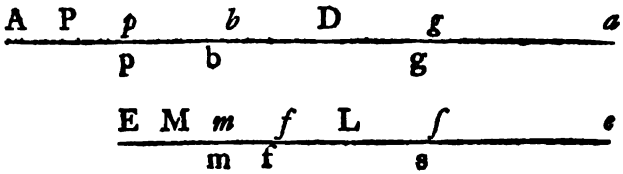
retarded, and the velocity of *m* is accelerated. For it appears, in the ſame manner, that the velocity of *m* at L is greater than the velocity of M at F, but leſs then the velocity of M at S; ſo that it muſt be equal to the velocity of M at ſome intermediate term of the ſpace FS: and it appears, from the firſt and ſecond axioms, that it cannot be equal to the velocity of M at any term of the ſpace FS, but L only.

58. The motion of M being ſtill accelerated continually, and the motion of *p* retarded continually, let the motion of *m* be alſo retarded. If the velocity of *m* at L be greater than the velocity of M at L, let the velocity of p in the line A*a* be greater than the velocity of P in the ſame ratio. The conſtant velocity of p being greater than the velocity of *p* at D, and the motion of *p* being retarded in a continued manner, the velocity of p ſhall be equal to the velocity of *p* at ſome term before it comes to



D, as at *b*. Then, Em being determined from Ap in the ſame manner as EM is determined from AP, the motion of m ſhall be continually accelerated, (art. 48.) and the velocity of m at any term of the line E*e*, as L, ſhall be to the velocity of M at the ſame term, as the velocity of p is to the velocity of P, (by the tenth theorem,) or as the velocity of *m* at L is to the velocity of M at L. Therefore the velocity of m at L muſt be ſuppoſed equal to the velocity of *m* at L. Let *p*, p, *m* and m deſcribe the ſpaces *b*D, bD, *f*L and fL in the ſame time; and bD being greater than *b*D, (by the third axiom, the velocity of p being equal to the velocity of *p* at *b*,) fL muſt be greater than *f*L. The point m, with its motion at L continued uniformly, would deſcribe a greater ſpace than fL in that time, (by the ſecond axiom.) The point *m*, with its motion at L continued uniformly, would deſcribe a leſs ſpace than *f*L (which is leſs than fL) in the ſame time, (by the fourth axiom.) Therefore the velocity of m at L is greater than the velo-

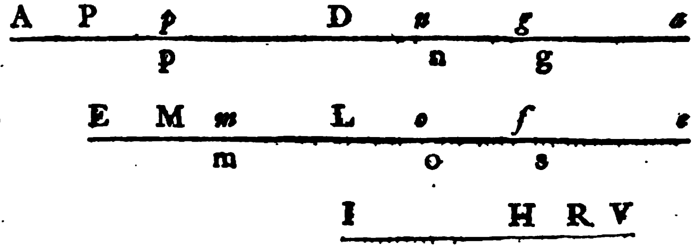
velocity of *m* at L. But theſe velocities were ſuppoſed equal; and theſe being contradictory, it follows, that the velocity of *m* at L is not greater than the velocity of M at L. If the velocity of *m* at L be leſs than the velocity of M at L, let the velocity of p be leſs than the velocity of P in the ſame ratio; and, ſince it is leſs than the velocity of *p* at D, let it be equal to the velocity of *p* at ſome ſubſequent term of the line A*a*, as at *g*. Em being determined from Ap in the ſame manner as EM is determined from



AP, the velocity of m at L is to the velocity of M at L, as the velocity of p is to the velocity of P, (by the tenth theorem;) and therefore muſt be ſuppoſed equal to the velocity of *m* at L. Let *p*, p, *m* and m deſcribe the ſpaces D*g*, Dg, L*ſ* and Ls in the ſame time; and Dg being leſs than D*g*, (by the fourth axiom,) Ls is leſs than L*ſ*. The point m, with its motion at L continued uniformly, would deſcribe a ſpace leſs than Ls in that time, (by the firſt axiom;) and the point *m*, with its motion at L continued uniformly, would deſcribe a greater ſpace than L*ſ* (which exceeds Ls) in the ſame time, (by the third axiom.) Therefore the velocity of m at L is leſs than the velocity of *m* at L. But they were ſuppoſed equal; and theſe being contradictory, it appears, that the velocity of *m* at L is neither greater nor leſs than the velocity of M at L, but preciſely equal to it.

59. The motion of M being accelerated continually, as formerly, let the motion of *p* be alſo accelerated continually, and it is evident that the motion of *m* is alſo accelerated, (art 14.) If the velocity of *m* at L be not equal to the velocity of M at L, let it firſt be greater, in the ſame ratio as IV is greater than IH; and (R being any point betwixt H and V) let the velocity of p be to the velocity of P (or the velocity of *p* at D) as IR is to IH; and, ſince it is greater than the velocity of *p* at D, ſuppoſe it equal to the velocity of *p* at *g*. Let p deſcribe Dg

Dg with its uniform motion, in the ſame time that *p* deſcribes D*g*, and Dg ſhall be greater than D*g*, (by the ſecond axiom.) Let Em be determined from Ap in the ſame manner as EM is determined from AP; and, if m and *m* deſcribe the ſpaces Ls and L*ſ* in that time, Ls ſhall be greater than L*ſ*. The point *m*, with its motion at L continued uniformly, would deſcribe a leſs ſpace than L*ſ* in this time, (by the firſt axiom.) The point m, with its motion at s continued uniformly, would deſcribe a greater ſpace than Ls in the ſame time, (by the ſecond axiom.) Therefore the velocity of *m* at L is leſs than the velocity of m at s: but it is greater than the velocity of m at L, (in the ſame proportion as IV is greater than IR;) and, conſequently, it muſt be equal to the velocity of m at ſome intermediate term of the ſpace Ls, as o. In the ſame time that m deſcribes Lo, let *p*, p and *m* deſcribe the ſpaces D*n*, Dn and L*o* reſpectively.



The point *m*, with its motion at L continued uniformly, would deſcribe a ſpace leſs than L*o* in that time, (by the firſt axiom.) The point m, with its motion at o continued uniformly, would deſcribe a ſpace greater than Lo in the ſame time, (by the ſecond axiom;) and, Dn being greater than D*n*, (becauſe the velocity of p always exceeds the velocity of *p* till *p* come to *g*,) Lo is greater than L*o*. Therefore the velocity of *m* at L is leſs than the velocity of m at o. But they were ſuppoſed equal; and theſe being contradictory, it appears, that the velocity of *m* at L is not greater than the velocity of M at L. In like manner it is demonſtrated, that the velocity of *m* at L is not leſs than the velocity of M at L; and therefore theſe velocities are equal to each other.

60. The other caſes of this theorem, when the motion of M is ſuppoſed to be retarded continually, are demonſtrated in the ſame

ſame manner: or they may be deduced from thoſe we have deſcribed, by the ninth theorem. When the motions begin or end at the terms D and L, the ſame demonſtration is applicable; ſince it is ſufficient, that the motions may be conceived to have begun before theſe terms, or to be continued after them. For the ſame reaſon, theſe demonſtrations may be applied, when L is a term where the motion of M ceaſes to be accelerated, being afterwards retarded; or where it ceaſes to be retarded, being afterwards accelerated. In like manner, the theorem may be extended to thoſe caſes, when the velocity of M is increaſed or diminiſhed at L by any finite or aſſignable quantity, by conceiving the velocity thus augmented or diminiſhed to have been produced by a continued acceleration or retardation while M came to L.

61. In general, it follows from what has been demonſtrated, that when the points P and *p* deſcribe the line A*a* with motions that are either uniform or varied continually; and, EM being determined from AP in any regular manner, E*m* is determined from A*p* in the ſame manner: then the velocity of *m*, at any term of the line E*e*, is to the velocity of M at the ſame term of that line, as the velocity of *p*, at the correſponding term of the line A*a*, is to the velocity of P at the ſame term.

62. In the two following theorems, when we ſay a ratio is a limit betwixt two other ratios, we mean no more, but that it is greater than the one, and leſs than the other.

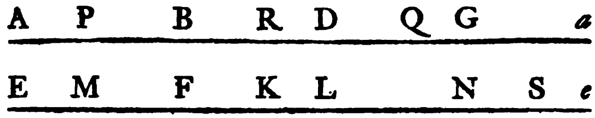
THEOREM XII.

The velocity of a motion that is accelerated or retarded perpetually, is, at any term of the time, to the velocity of an uniform motion, in a ratio that is always a limit between the ratio of the ſpaces deſcribed by theſe motions in any equal times before that term, and the ratio of the ſpaces deſcribed by them in any equal times after it.

While the point P deſcribes the line A*a* with an uniform motion,

tion, let the point M deſcribe the line E*e* with a motion that is accelerated or retarded perpetually. When P comes to D, let M come to L. Let BR and FK be ſpaces deſcribed by the points P and M in any time before they come to D and L; and let QG and NS be ſpaces deſcribed by them in any time after that term: and the velocity of M at L ſhall be to the conſtant velocity of P, in a ratio that is always a limit betwixt the ratio of FK to BR and the ratio of NS to QG.

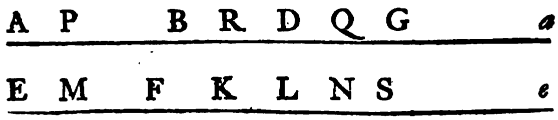
Firſt, let the motion of M be accelerated; and the point M, with its motion at N continued uniformly, would deſcribe a ſpace leſs than NS, in the ſame time the point P with its uniform motion deſcribes QG, (by the firſt axiom.) Therefore the velocity of the



point M at N is to the conſtant velocity of the point P (art. 4.) in a leſs ratio than NS is to QG. But the motion of the point M being perpetually accelerated, its velocity at L is leſs than its velocity at N, and therefore is to the conſtant velocity of P in a leſs ratio than that of NS to QG. By the ſecond axiom, the point M, with its motion at K continued uniformly, would deſcribe a greater ſpace than FK, in the ſame time P with its uniform motion deſcribes the ſpace BR; and therefore the velocity of the point M at K, is to the conſtant velocity of the point P in a greater ratio than FK is to BR. But the velocity of the point M at L is greater than its velocity at K; and therefore is to the conſtant velocity of P in a greater ratio than FK is to BR. Thus it appears, that the velocity of the point M at L is to the conſtant velocity of P, in a ratio that is always a limit betwixt the ratio of NS to QG and the ratio of FK to BR; being in this caſe leſs than the former, and greater than the latter of thoſe ratios.

63. If the motion of the point M be retarded perpetually, then (by the third axiom) the point M, with its motion at N continued uniformly, would deſcribe a greater ſpace than NS, in the ſame time P with its uniform motion deſcribes the ſpace QG; and therefore the velocity of M at N is to the con-

ſtant velocity of P in a greater ratio than NS is to QG. But the motion of M being now retarded, its velocity at L is greater than its velocity at N; and therefore its velocity at L is to the velocity of P in a greater ratio than NS is to QG. By the fourth axiom, the point M, with its motion at K continued uniformly, would deſcribe a ſpace leſs than FK, in the ſame time P deſcribes BR with its uniform motion; and



therefore the velocity of M at K is to the conſtant velocity of P in a leſs ratio than FK is to BR. But the velocity of M at L is leſs than its velocity at K; and, conſequently, the velocity of M at L is to the conſtant velocity of P in a leſs ratio than FK is to BR. Therefore the velocity of M at L is to the conſtant velocity of P, in a ratio that is always a limit betwixt the ratio of NS to QG and the ratio of FK to BR, being in this caſe greater than the former, and leſs than the latter of thoſe ratios. It is evident, that in either caſe the velocity of M at L is to the velocity of P, in a ratio that is always a limit betwixt the ratio of KL to RD, and that of LN to DQ; for this is only a particular caſe of the theorem.

THEOREM XIII.

64. *The ſpace deſcribed by a motion that is accelerated or retarded perpetually, is to the ſpace deſcribed in the ſame time by an uniform motion, in a ratio that is a limit betwixt the ratio of the velocities of theſe motions at the beginning of the time and their ratio at the end of it.*

The points P and M being ſuppoſed to deſcribe the ſpaces DG and LS in the ſame time, and the motion of M being accelerated, as in the 62d article; then, ſince the point M, with its motion at L continued uniformly, would deſcribe a leſs ſpace than

than LS, in the time P defcribes the fpace DG with an uniform motion, (by the firft axiom;) it follows, that the fpace LS is to the fpace DG in a greater ratio, than the velocity of M at L is to the conftant velocity of P. The point M, with its motion at S continued uniformly, would defcribe a greater fpace than LS, in the time P defcribes DG with its uniform motion, (by the fecond axiom;) and, confequently, the fpace LS is to the fpace DG in a lefs ratio, than the velocity of M at S is to the conftant velocity of P. Therefore the ratio of the fpace LS to the fpace DG, is a limit betwixt the ratio of the velocity of M at L to the conftant velocity of P, and the ratio of the velocity of M at S to the velocity of P; being greater than the former, and lefs than the latter of thofe ratios.

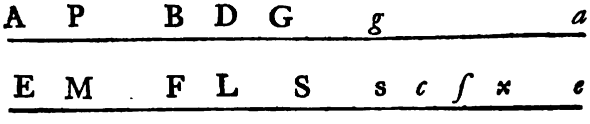
65. Let the motion of the point M be perpetually retarded, as in the 63d article; and, by the third axiom, the point M would defcribe a greater fpace than LS, with its motion at L continued uniformly, in the time P defcribes DG; but the point M, with its motion at S continued uniformly, would defcribe a lefs fpace than LS in the time P defcribes DG, (by the fourth axiom:) Therefore the fpace LS is to the fpace DG in a ratio that is lefs than the ratio of the velocity of M at L to the velocity of P, but greater than the ratio of the velocity of M at S to the velocity of P.

THEOREM XIV.

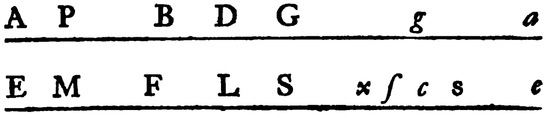
66. *The motion of the point* P *being uniform, but the motion of the point* M *continually varied, let the velocity of* P *be to the velocity of* M *at* L, *as a given line* Dg *is to* Lc; *let* Dg *be always to* Lf, *as the fpace* DG *defcribed by* P *in any time, is to* LS *the fpace defcribed by* M *in the fame time. Then, by diminifhing the fpaces* DG *and* LS *continually,* cf *may become lefs than any affignable magnitude.*

Let *cx* be any fmall quantity affigned at pleafure; and let it be added to L*c* when the motion of M is accelerated, but fub-

ducted from L*c* when the motion of M is retarded: and it is manifest, that, by diminishing LS, the velocity of M at S approaches continually to the velocity of M at L, so that their difference may become equal to the difference of any two unequal velocities that can be assigned, or less than it, how small soever it may be. Let LS be diminished till the difference of those velocities be to the constant velocity of P as *cx* is to D*g*; and the velocity of M at S shall be to the velocity of P as L*x* is to D*g*.



When the motion of M is accelerated, it follows, from the 64th article, that the ratio of LS to DG, or of L*ſ* to D*g*, is greater than the ratio of the velocity of M at L to the velocity of P, or the ratio of L*c* to D*g*; but that the same ratio of LS to DG, or of L*ſ* to D*g*, is less than the ratio of the velocity of M at S to the velocity of P, or the ratio of L*x* to D*g*. Therefore L*ſ* is greater than L*c*, but less than L*x*; and, consequently, *cſ* is less than *cx*. When the motion of M is retarded continually, then (by the 65th article) the ratio of LS to DG, or of L*ſ* to D*g*, is less than the ratio of the velocity of M at L to the velocity of P,



or the ratio of L*c* to D*g*, but greater than the ratio of the velocity of M at S to the velocity of P, or the ratio of L*x* to D*g*. Therefore L*ſ* in this case is less than L*c*, but greater than L*x*; and, consequently, *cſ* is less than *cx*. In the same manner it is demonstrated, that if FL be always to BD as Ls is to D*g*, then, by diminishing the spaces FL, BD, which are described by M and P before they come to L and D, *c*s may become less than any given magnitude. And if FL, LS be spaces described by the point M in equal times, or in times that are to each other in any given proportion; and FL, LS, DG be to each other always in the same proportion as Ls, L*ſ* and D*g*: then, by diminishing the spaces FL and LS continually, s*ſ* may become less

leſs than any given magnitude. It appears from what was ſhewn in the 44th article, that any motion muſt be ſuppoſed to be either uniform, or varied in a continued manner for ſome time, how ſmall ſoever that time may be; and therefore this theorem obtains univerſally.

67. Becauſe *cſ* the difference betwixt L*ſ* and L*c* decreaſes ſo that it may become leſs than any given quantity, how ſmall ſoever, when DG and LS are diminiſhed continually; it appears, that the ratio of D*g* to L*ſ* (or of DG to LS) approaches continually to the ratio of D*g* to L*c*, ſo that it may come nearer to this ratio, than the ratio of D*g* to any aſſignable quantity greater or leſs than L*c*. For this reaſon, the ratio of L*c* to D*g* is by Sir Isaac Newton called the *Limit* of the variable ratio of L*ſ* to D*g*, or of LS to DG, in a more reſtricted ſenſe of this term than that in which we made uſe of it in the twelfth and thirteenth theorems.

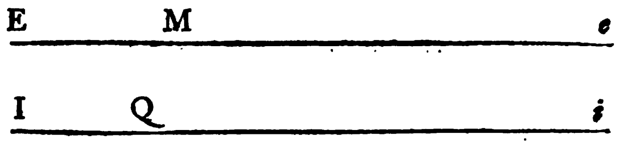
68. When the motion of the point M is continually accelerated from L to S, then L*ſ* conſiſts always of two parts: the part L*c* is invariable, and meaſures the velocity of M at L; the part *cſ* is variable, and ariſes from the acceleration of the motion of M while it deſcribes LS. This latter part decreaſes continually when DG and LS are diminiſhed, and vaniſhes with them. Therefore, when EM is determined from AP by any conſtruction or equation, and thence the variable ratio of LS to DG, or of L*ſ* to the given quantity D*g*, is reduced to a rule or expreſſion, all that is requiſite to determine the ratio of L*c* to D*g* is, to diſtinguiſh betwixt L*c* the invariable part of L*ſ* and the variable part *cſ*. And, for this purpoſe, it is ſufficient to ſuppoſe DG and LS to decreaſe, and to find what part of L*ſ* continually decreaſes at the ſame time, and at length vaniſhes with LS; for this part is *cſ*: which being rejected, the remainder L*c* is to D*g* as the velocity of M at L is to the conſtant velocity of P, or as the fluxion of EL is to the fluxion of AD. When the motion of M is continually retarded, then L*ſ* is leſs than L*c* by the difference *cſ* which decreaſes and vaniſhes with LS, as before; and this part of the expreſſion of L*ſ* being diſcovered and rejected in the ſame manner, the other part gives L*c*, which is to the given line D*g* as the fluxion of EL is to the fluxion of AD.

69. It

69. It is in this concise manner Sir ISAAC NEWTON most commonly determines the ratio of the fluxions of quantities. But we shall treat more fully of his method afterwards; and, since there have been various objections made against this doctrine, we shall demonstrate its principal propositions immediately from the axioms. By tracing them to such plain principles, their evidence may be more easily examined, and objections against them may either be obviated, or, if any doubt or difficulty remain, it may appear wherein precisely it lyes. It is worth while to demonstrate the chief propositions of this method in as clear and compleat a manner as possible, if by this means we can preserve this science from disputes. Some of the preceeding theorems are so evident, that they are commonly admitted without a proof: but, because we are delivering the elements of this doctrine, and have proposed, in treating it, to imitate the ancient Geometricians, (who never increased the number of their principles without necessity,) we have deduced those theorems from the axioms; that it might appear how few and plain the principles are which it is necessary for us to assume in demonstrating it. It remains, before we proceed to enquire into the fluxions of particular quantities, that we should say something of the higher orders of fluxions.

70. When a motion is accelerated or retarded continually, the velocity may be itself considered as a variable or flowing quantity, and may be represented by a line that increases or decreases continually. When a velocity increases uniformly, so as to acquire equal increments in any equal times, its fluxion is measured by the increment which is generated in any given time. In this case, the velocity is represented by a line that is described with an uniform motion; and its fluxion, by the constant velocity of the point that describes the line, or by the space which this point describes in a given time. When a velocity is not accelerated uniformly, but acquires increments in equal times that continually increase or decrease, then its fluxion at any term of the time is not measured by the increment which it actually acquires, but by that which it would have acquired if its acceleration had been continued uniformly from that term for a given time. And, in the same manner, when a motion is retarded con-

continually, the quantity by which it would be diminiſhed in a given time, if its retardation was continued uniformly from any term, meaſures its fluxion at that term. While the point M deſcribes the line Ee, let the point Q deſcribe the line Ii, ſo that IQ may be always equal to the ſpace that would be deſcribed by the motion of M if it was continued uniformly for a given time. Then IQ ſhall always repreſent the velocity of M, and the velocity of the point Q ſhall repreſent the fluxion of the velocity of M;



which therefore is meaſured, at any term of the time, by the ſpace which would be deſcribed by Q with its motion at that term continued uniformly for a given time. The velocity of M is the fluxion of EM; and therefore the velocity of Q repreſents the fluxion of the fluxion of EM. Thus, when a fluxion of a quantity is variable, it may be conſidered itſelf as a fluent, and may have its fluxion, which is called the *Second fluxion* of that quantity. This may alſo have its fluxion, which is called the *Third fluxion* of the firſt fluent: and we ſhall ſhew afterwards, that motions may be eaſily conceived to vary in ſuch a manner, as to give ground for admitting ſecond fluxions, and thoſe of any higher order.

71. The ſecond fluxions are deduced from the firſt, in the ſame manner and upon the ſame principles as the firſt fluxions are deduced from their fluents: and therefore we ſhall ſubjoin in this place but one theorem concerning them.

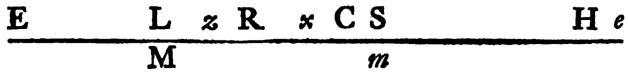
LEMMA II.

When a motion is accelerated or retarded uniformly, the ſpace deſcribed by it is an arithmetical mean betwixt the ſpaces that would be deſcribed in the ſame time by the motions at the beginning and end of that time continued uniformly.

Let

Let the point M deſcribe the line LS in any given time, with a motion that is accelerated or retarded uniformly; let LC and SH be the ſpaces that would be deſcribed in an equal time by its motions at L and S continued uniformly. Then the difference of SH and LS ſhall be equal to the difference of LS and LC.

Let the point *m* move from S to L, deſcribing always any ſpaces upon SL in times equal to thoſe in which they are deſcribed by M, but in a contrary order, (as in art. 47.) and the velocity of *m* at any term of the line LS, muſt be equal to the velocity of M when it comes to the ſame term of that line, by the ninth theorem. In the ſame time the point M deſcribes any ſpace L*z*, let *m* deſcribe S*x*; and, ſince the time in which M deſcribes *x*S is equal to the time in which *m* deſcribes it, (by the ſuppoſition,) it follows, that L*z* and *x*S are deſcribed by M in equal times. Therefore, ſince the motion of M increaſes or decreaſes uniformly, the difference of its velocities



at L and *z* is equal to the difference of its velocities at *x* and S; and, conſequently, is equal to the difference of the velocities of *m* at *x* and S. From which it follows, that the ſum of the velocities of M and *m*, is always equal to the ſum of the velocities of M at L and of *m* at S, or to the ſum of the velocity of M at L added to its velocity at S. Therefore, by the eighth theorem, LS the ſpace deſcribed by M added to LS the ſpace deſcribed by *m* in the ſame time, is equal to the ſum of the ſpaces LC and SH that would be deſcribed in an equal time by the motions of M at L and S continued uniformly; and, conſequently, the difference of SH and LS is equal to the difference of LS and LC.

72. Let LR and RS be any ſpaces deſcribed by the point M with a motion that is accelerated or retarded uniformly, in equal times that immediately ſucceed after one another; and, in the ſame time that M deſcribes the ſpace LS with this motion, it would deſcribe a ſpace equal to LS by its motion at R continued uniformly. For the velocity of M at R is an arithmetical mean betwixt its velocities at L and S, becauſe the motion of

of M increaſes or decreaſes uniformly; and therefore the point M, with its motion at R continued uniformly, would deſcribe a ſpace equal to half the ſum of LC and SH, in the ſame time that it would deſcribe LC with its motion at L, or SH with its motion at S continued uniformly. But, by this lemma, LS is equal to half the ſum of LC and SH; and the point M deſcribes LS with its accelerated motion in the ſame time that it would deſcribe LC with its motion at L continued uniformly. Therefore, in the ſame time that the point M deſcribes LS with a motion uniformly accelerated, it would deſcribe a ſpace equal to LS with its motion at R continued uniformly.

73. If the motion of the point M begin at the term L from nothing, then LC vaniſhes, and LS is equal to one half of SH; that is, when the motion begins from nothing, and is accelerated uniformly for any time, the ſpace deſcribed by it is one half of the ſpace deſcribed in an equal time by the motion that is acquired by this acceleration continued uniformly. This is one of the propoſitions diſcovered by GALILEUS; and ſeveral others of this kind may be demonſtrated in the ſame manner from the preceeding theorems, without having recourſe to the method of indiviſibles or of infiniteſimals.

THEOREM XV.

74. *Let the point* M *deſcribe the line* Ee *with any variable motion; and, in the ſame time that it would deſcribe* LC *with its motion at* L *continued uniformly, ſuppoſe that it would deſcribe* LS *if the acceleration or retardation of its motion was continued uniformly from that term. Then, if the velocity of* M *at* L, *or the firſt fluxion of* EL, *be repreſented by* LC, *the ſecond fluxion of* EL *may be meaſured by* 2CS.

The fluxion of the velocity of M, at any term of the time of its motion, is meaſured by the increment which it acquires in a given time when its acceleration is continued uniformly

from that term. The acceleration of the motion of M being ſuppoſed to be continued uniformly, till it deſcribe LS in the ſame time that it would deſcribe LC with its motion at L continued uniformly; let SH be to LC as the velocity which M would acquire in this manner at S is to its velocity at L: and, ſince the velocity of M at L is ſuppoſed to be repreſented by LC, the velocity which it would thus acquire at S will be re-

E M F L C S *c* *ſ* H *e*

I Q K V *v* *i*

preſented by SH; and the increment of its velocity which would be generated if the acceleration of its motion was continued uniformly from L, in the ſame time that it would deſcribe LC with its motion continued uniformly from L, will be repreſented by the exceſs of SH above LC; which is equal to 2CS, by the laſt lemma. Therefore the fluxion of the velocity of M at L, or the ſecond fluxion of EL, is repreſented by 2CS. When the motion of M is retarded, then the decrement of the velocity of M that would be produced if the retardation of its motion was continued uniformly from L, in the ſame time in which M would deſcribe LC with its motion continued uniformly from L, is repreſented by the exceſs of LC above SH; and the ſecond fluxion of EL is repreſented by that exceſs, or by 2CS.

75. Or if we ſuppoſe, as in the 70th article, that the line IQ always repreſents the velocity of M, or the fluxion of EM, the velocity of the point Q will repreſent the fluxion of the velocity of M, or the ſecond fluxion of EM. Becauſe LC is ſuppoſed to repreſent the velocity of M at L, let IK be equal to LC, and Q ſhall come to K when M comes to L. The velocity of Q at K is meaſured by the ſpace which would be deſcribed by its motion at K continued uniformly, in the ſame time that M, with its motion at L continued uniformly, would deſcribe LC. Let KV be this ſpace; and, the motion of Q being ſuppoſed uniform while it deſcribes KV, the acceleration or retarda-

retardation of the motion of M is therefore continued uniformly for that time; and the velocity of M at S is to its velocity at L, as IV is to IK or LC. Therefore the point M would describe ſpaces equal to IK and IV by its motions at L and S continued uniformly, in the ſame time that it deſcribes LS when the acceleration or retardation of its motion is continued uniformly; and, by the laſt lemma, KV is equal to 2CS. But KV repreſents the velocity of Q at K, or the ſecond fluxion of EL; which is therefore alſo repreſented by 2CS. When the motion of M is retarded, IQ decreaſes, the point Q moves from K towards I, and the ſecond fluxion of EM is in this caſe ſaid to be negative, being conſidered as a power that retards the generating motion, diminiſhing continually the firſt fluxion of EM. It appears from this theorem, that, as the firſt fluxion of a variable quantity, at any term of the time, is meaſured by the increment or decrement which would be produced if the generating motion was continued uniformly from that term for a given time; ſo its ſecond fluxion may be meaſured by twice the difference betwixt this increment or decrement, and that which would be produced if the acceleration or retardation of the generating motion was continued uniformly from that term for the ſame time.

76. Let Kv, Lc and L$ſ$ be any other ſpaces that would be deſcribed in the ſame time by the uniform motion of Q, the motion of M at L continued uniformly, and the motion of M if its acceleration was continued uniformly from that term, reſpectively. Then the velocity of M acquired by this laſt motion at $ſ$ ſhall be to its velocity at L, as Iv is to IK or LC: and, the difference of thoſe velocities being to the velocity of M at L as 2$cſ$ is to Lc, by the laſt lemma; it follows, that 2$cſ$ is to Kv as Lc is to LC. But Kv is to KV as Lc is to LC, (by the ſecond theorem;) and therefore $cſ$ is to CS in the duplicate ratio of Lc to LC. It is alſo evident, that when LC and LS are ſuppoſed to decreaſe continually, the ratio of CS to LC decreaſes ſo that it may become leſs than any aſſignable ratio. For the ratio of the velocity of M at S to its velocity at L, or that of SH to LC, approaches continually to a ratio of equality, as the point S approaches to L, and may come nearer to it than

any aſſignable ratio of inequality. But CS is leſs than the difference of SH and LC; and therefore the ratio of CS to LC may become leſs than any aſſignable ratio. From this it follows, that when LC, which repreſents the firſt fluxion of EM, continually decreaſes, then 2CS, which repreſents its ſecond fluxion, decreaſes ſo that its ratio to LC may become leſs than any aſſignable ratio.

77. Let FL and LS be ſpaces deſcribed by the point M in any equal times that ſucceed after one another; and let KV be deſcribed by Q in the ſame time M deſcribes LS. When the motion of M is accelerated or retarded uniformly, its velocity at L, or the firſt fluxion of EL, may be meaſured by half the ſum of FL and LS; and the ſecond fluxion of EL may be meaſured by the difference of LS and FL. For the point M would deſcribe a ſpace equal to half the ſum of FL and LS by its motion at L continued uniformly, in the ſame time that it deſcribes LS with a motion accelerated or retarded uniformly, by the 72d article; and KV, or 2CS, (which meaſures the ſecond fluxion of EL when LC meaſures its firſt fluxion,) is equal to the difference of the ſpaces LS and FL. When the motion of M is accelerated uniformly, the ſpace LS which is deſcribed by M in a given time, is equal to LC, that repreſents the firſt fluxion of EL, added to the half of KV that repreſents the ſecond fluxion of EL; and LS is equal to the difference of LC and one half of KV when the motion of M is retarded uniformly. In other caſes, when the acceleration or retardation of the motion of M is not uniform, but increaſes or decreaſes continually while it deſcribes the ſpace FS, LC is not equal to half the ſum of FL and LS; but it follows from the 66th and 67th articles, that its ratio to half their ſum approaches continually to a ratio of equality as its limit, when thoſe ſpaces are continually diminiſhed. And in the ſame manner it appears, that, by diminiſhing LC, the ratio of KV to the difference of LS and FL, and the ratio of LS to the ſum or difference of LC and one half of KV, continually approach to a ratio of equality, ſo that they may come nearer to it than any aſſignable ratio of inequality. The ratio of *cſ* to CS (the differences betwixt the ſpaces deſcribed by M and thoſe which would be deſcribed in the ſame times by its motion conti-

continued uniformly from L) approaches continually to the duplicate ratio of L*c* to LC, by diminiſhing thoſe ſpaces, ſome caſes excepted that will be deſcribed afterwards. There are theorems analogous to theſe which relate to the higher orders of fluxions; but we ſhall demonſtrate them afterwards, and proceed now to enquire into the fluxions of geometrical magnitudes. Beſides the preceeding general theorems, there are others concerning the compoſition and reſolution of motion, which are ſometimes conſidered as the grounds of this method: but they may rather ſerve for applying this general doctrine to particular caſes; and therefore we refer them to another place.

CHAP. II.

Of the Fluxions of plane rectilineal Figures.

PROPOSITION I.

78. *The fluxion of a parallelogram of an invariable altitude, is always meaſured by a parallelogram of the ſame altitude deſcribed upon the right line which meaſures the fluxion of the baſe.* Fig. 16. Pl. 3.

While the point P deſcribes the baſe AO, let the given right line PM, by moving parallel to itſelf, generate the parallelogram APMF. When P comes to D, let PM come to DE; and, if the fluxion of the baſe at that term of the time be repreſented by DG, the fluxion of the parallelogram at the ſame term ſhall be repreſented by the parallelogram EG.

When the motion of P is uniform, (that is, when it deſcribes equal ſpaces in any equal times,) the right line PM deſcribes equal parallelograms in equal times, (Elem. 36. 1.) Therefore the motion of PM is alſo uniform, (art. 14.) and in the ſame time that the point P deſcribes DG with an uniform motion, the right line PM deſcribes the parallelogram EG with an uniform motion. Therefore, the fluxion of the baſe AD being repreſented

sented by DG, the fluxion of the parallelogram AE is represented by the parallelogram EG, by art. 11.

79. Since P is a point or term of the line PM, which is of a given or invariable magnitude, and is supposed to move always parallel to itself, any point in this line moves with the same velocity as P. According as the motion of P is accelerated or retarded, the motion of the right line PM is accelerated or retarded; and, in the same time that the point P would describe DG with its motion at D continued uniformly, the right line PM would describe the parallelogram EG with its motion continued uniformly from the same term. If this can be supposed to need any other proof, it may be demonstrated from the axioms in the following manner.

80. When the motion of P is continually accelerated, the motion of PM is also accelerated continually, (Elem. 1. 6. and art. 14.) Let the point *p* describe the base with an uniform motion equal to that of P at D; let *pm*, equal and parallel to PM, generate the parallelogram A*pm*F: and the constant velocity of *pm* shall be equal to the velocity of PM at the term or moment when P comes to D. For, while *p* with its uniform motion describes any spaces *g*D and DG, let P with its accelerated motion describe the spaces *k*D and DK; and, since the velocity of P at D is equal to the constant velocity of *p*, DK is greater than DG, (by ax. 1.) and D*g* greater than D*k*, (by ax. 2.) Complete the parallelograms EG, E*g*, EK, E*k*; and EK shall be greater than EG, but E*k* less than E*g*. By ax. 2. the right line PM would describe a greater space than EK, with its motion at the term K continued uniformly, in the same time P describes DK, or *pm* describes EG. Therefore the velocity of PM at K is greater than the constant velocity of *pm*. By ax. 1. the right line PM, with its motion at *k* continued uniformly, would describe a less space than E*k* in the time P and *p* describe *k*D and *g*D; and in the same time *pm* with a constant velocity describes E*g*, which is greater than E*k*. Therefore the velocity of PM at *k* is less than the constant velocity of *pm*. In the same manner it is demonstrated, that the velocity of *pm* is less that the velocity of PM at any term after P passes D, but is greater than the velocity of PM at any term before

P

P comes to D. Therefore the velocity of *pm* is equal to the velocity of PM at the term or moment when P comes to D. When the motion of P is retarded, it appears in the ſame manner, from ax. 3. & 4. that the motion of PM at D is equal to the conſtant velocity of *pm*. The uniform velocity of *p*, or the motion of P at D, being meaſured by DG, the motion of *pm* is meaſured by the parallelogram EG, (by art. 78.) which therefore meaſures the motion of PM at D, or the fluxion of the parallelogram AM when AM becomes equal to AD. In the ſame manner it is demonſtrated in general, that when a given line, by revolving about a given center or axis, deſcribes any area; or when a given ſurface, by moving parallel to itſelf, or by revolving on a given axis, generates a ſolid: the motion with which the area or ſolid flows at any given term is always the ſame, when the velocity of the generating figure at that term is the ſame, whatever variation the motion of the generating figure may be ſubject to before or after that term.

81. Let the right lines AO, AV be given in poſition; and, Fig. 17. while the point P deſcribes the baſe AO, let the right line PM, by moving parallel to itſelf, generate the triangle APM. At the ſame time, let the point *p* deſcribe the right line *ao*; and a given or invariable right line *pm*, by moving parallel to itſelf, generate a parallelogram *am* always equal to the triangle APM. When P comes to D, let *p* come to *d*, PM to DE, and *pm* to *de*: Then the velocity with which the baſe AD flows, is the ſame as the velocity of the point P at the term or moment when it comes to D, (art. 10.) and the velocity with which the triangle ADE flows at that term, is the ſame as the velocity of the invariable right line *pm* when *p* comes to *d*. Therefore, when the velocity of P is given at any term of the time, to determine thence the velocity of the right line *pm* at that term, is the ſame as from the fluxion of the baſe AP to determine the fluxion of the triangle APM, (art. 11.) and if the velocity of *pm* is given, to determine thence the velocity of P, is the ſame as from the fluxion of the triangle APM to determine the fluxion of the baſe AP.

82. It is manifeſt, that when the place of the point P at any term of the time is given, the place of the right line *pm* at that

term.

term is found by applying upon the given right line *af* a parallelogram *am* equal to the triangle APM, (the angle *mpo* being always supposed equal to MPO.) If the place of *pm* is given, and that of P is required; let AO be of such a magnitude that OV parallel to PM may be equal to 2*af* or 2*pm*; and take AP a mean proportional betwixt *ap* and AO. For if *ap* be to AP as AP is to AO, or as PM is to OV, (which is equal to 2*af*;) the rectangle contained by *ap* and *af* shall be equal to half the rectangle contained by AP and PM, and the parallelogram *am* shall be equal to the triangle APM.

LEMMA III.

83. *When the base increases uniformly, the triangle increases with a motion that is perpetually accelerated; but when the base decreases uniformly, the triangle decreases with a motion that is perpetually retarded.*

The same things being supposed as in the two last articles, the motion of the point P is the motion with which the base flows; and the motion of the given right line *pm* is the same as that with which the triangle flows. Let the right line *pm* describe the parallelograms *be* and *db* in any equal times that immediately succeed after one another; and let the point P describe the right lines BD and DG in the same equal times. Let BC, DE and GH, parallel to PM, meet AP in C, E and H. Then, because the spaces described by *pm* are supposed to be always equal to the spaces described in the same time by PM, the parallelogram *be* will be equal to the trapezium BDEC, and the parallelogram *db* equal to the trapezium DGHE. Because the motion of P is uniform, BD is equal to DG, and the trapezium DGHE is greater than BDEC in the same proportion as the sum of DE and GH is greater than the sum of DE and BC, or the sum of AG and AD is greater than the sum of AD and AB; and the parallelogram *db* is greater than *be* in the same proportion. Therefore, when the base increases uniformly, the spaces *be* and

and *db* described by *pm* in any equal times that succeed after one another, perpetually increase; and the motion of *pm* is accelerated perpetually. But when the base decreases, the point P moves from O towards A; the right line *pm*, by moving from *o* towards *a*, describes in any equal times that succeed after one another the parallelograms *bd* and *eb* that perpetually decrease; and its motion in this case is perpetually retarded. The motion with which the triangle flows, is measured by the motion of *pm*; and is therefore perpetually accelerated when the base increases uniformly, but perpetually retarded when the base decreases uniformly.

84. When the triangle APM increases uniformly, the base increases with a motion that is perpetually retarded; but when the triangle decreases uniformly, the base decreases with a motion that is perpetually accelerated. For, when the triangle APM increases or decreases uniformly, the motion of the right line *pm* is uniform, by the supposition. While *pm* describes the equal parallelograms *be* and *db* in any equal times, let the point P describe the right lines BD and DG; and, the trapezium BDEC being equal to the parallelogram *be*, and DGHE equal to the parallelogram *db*, the trapezium BDEC is equal to the trapezium DGHE, and BD is greater than DG in the same proportion as the sum of DE and GH is greater than the sum of DE and BC. Therefore, when the triangle increases uniformly, or the motion of *pm* from *a* towards *o* is uniform, the spaces BD and DG described by the point P in any equal times perpetually decrease, and its motion is perpetually retarded. But, when the triangle decreases uniformly, or the motion of *pm* from *o* towards *a* is uniform, the spaces GD and DB described by P in any equal times perpetually increase, and its motion is accelerated perpetually. Fig. 18.

85. All the rules for the operations in the direct method of fluxions may be deduced from the two following propositions; and there can hardly remain any ground for objections against it when these are established in an unexceptionable manner. We shall therefore demonstrate them at some length, by the method which seems to set the evidence of this doctrine in the clearest light, and to resolve in the most satisfying manner the difficulties that have been raised against its truth or accuracy.

PROP. II.

FIG. 17. & 19. *The ſides* AD, AE *of the triangle* ADE *being given in poſition, and the angle* ADE *being alſo given; in the ſame time that the motion with which the baſe* AD *flows, continued uniformly, would generate any right line* DG, *the motion with which the triangle* ADE *flows, continued uniformly, would generate the parallelogram* EG. *Or: The fluxion of the baſe* AD *being repreſented by* DG, *the fluxion of the triangle* ADE *is accurately meaſured by the parallelogram* EG.

While the point P deſcribes the baſe AO, and the variable right line PM by moving parallel to itſelf generates the triangle APM, let the invariable right line *pm*, by moving parallel to itſelf along the right line *ao*, generate the parallelogram *am* always equal to the triangle APM, ſo that the ſpaces deſcribed by *pm* may be always equal to thoſe deſcribed in the ſame time by PM, as in the preceeding articles: Then the motion with which the baſe flows, or its fluxion, ſhall be always meaſured by the velocity of the point P; and the motion with which the triangle APM flows, or its fluxion, ſhall be always meaſured by the velocity of the invariable line *pm*. When AP becomes equal to AD, let *ap* become equal to *ad*, (that is, let *p* come to *d* when P comes to D;) and the right lines PM, *pm* ſhall come to DE and *de* at the ſame term of the time. Suppoſe that, if the motion of P was continued uniformly from this term, it would deſcribe the line DG in any given time; and that, if the motion of *pm* was continued uniformly from the ſame term, it would deſcribe a ſpace equal to the parallelogram *ek*, in the ſame given time: Then ſhall the parallelogram *ek* be equal to the parallelogram EG.

FIG. 17. *Caſe* 1. When the baſe increaſes uniformly, or the motion of the point P from A towards O is uniform; the motion of *pm* from

from *a* towards *o* is a motion perpetually accelerated, by the laſt lemma. Let the point P deſcribe BD and DG in equal times that ſucceed immediately after each other; and let *pm* deſcribe the parallelograms *be* and *dh* in the ſame equal times. Let BC, DE and GH parallel to PM meet AV in C, E and H; and the parallelogram *be* ſhall be equal to the trapezium BDEC, and *dh* equal to DGHE, by the ſuppoſition. The motion of *pm* being accelerated perpetually, it follows from ax. 1. that the parallelogram *ek* is leſs than the parallelogram *dh*; becauſe the ſpace *dh* is deſcribed by *pm* with an accelerated motion, and *ek* is the ſpace that would be deſcribed in the ſame time by *pm* with its motion continued uniformly from the beginning of that time without any acceleration. By ax. 2. the ſame parallelogram *ek* is greater than *be*, which was deſcribed by *pm* in an equal time before its velocity at the term when it comes to *d* was acquired. Therefore the parallelogram *ek* is leſs than the trapezium DGHE, but greater than the trapezium BDEC. It is evident alſo, that the parallelogram EG is leſs than DGHE, but greater than BDEC. I ſay further, that the parallelogram *ek* is preciſely equal to EG. For, if it is not equal to EG, it muſt be greater or leſs than it. Let *ek* firſt be greater than EG, and produce DE beyond E to R, till DR be greater than DE in the ſame ratio; and, completing the parallelogram DRLG, it ſhall be to EG as DR is to DE, (Elem. 1. 6.) or as *ek* is to EG, (by the ſuppoſition:) and therefore the parallelogram RG will be equal to *ek*. Let RL meet CH in N, and NQ parallel to DE meet the baſe in Q. Suppoſe that *pm* would deſcribe the parallelogram *ex*, by its motion continued uniformly from the term when *p* comes to *d*, in the ſame time P deſcribes DQ with its uniform motion. Then, the ſpaces deſcribed by any uniform motion being in the ſame proportion as the times in which they are deſcribed, (by theor. 1. art. 16.) the parallelogram *ek*, or RG, ſhall be to the parallelogram *ex* as DG is to DQ, or as RG is to RQ; and therefore the parallelograms *ex* and RQ are equal. But, while the point P deſcribes DQ, the right line *pm* deſcribes a ſpace equal to the trapezium DENQ, by the ſuppoſition; and, its motion being perpetually accelerated during this time, it follows from ax. 1. that DENQ is greater

than *ex* the ſpace which would have been deſcribed in the ſame time by *pm* if its motion had been continued uniformly from the beginning of that time without any acceleration. And, ſince DR is greater than DE in the ſame proportion as *ek* is ſuppoſed greater than EG, the parallelogram RQ is greater than the trapezium DENQ; and therefore is ſurely greater than *ex*. But RQ was proved equal to *ex*: And theſe being contradictory, it follows, that the parallelogram *ek* is not greater than the parallelogram EG.

86. Let us ſuppoſe now that the parallelogram *ek* is leſs than the parallelogram EG; and, D*r* being ſuppoſed leſs than DE in the ſame proportion, complete the parallelogram *r*G; and, *r*G being to EG as D*r* is to DE, (Elem. 1. 6.) or as *ek* is to EG, *r*G muſt be equal to *ek*. Let *rl* produced meet CE in *n*, and *nq* parallel to DE meet the baſe in *q*. Suppoſe that *pm* would deſcribe the parallelogram *ex*, by its motion continued uniformly from the term when *p* comes to *d*, in a time equal to that in which P deſcribes *q*D. Then, by theor. 1. the parallelogram *ek*, or *r*G, ſhall be to the parallelogram *ex* as DG is to *q*D, or as the parallelogram *r*G is to *rq*; and therefore *ex* is equal to *rq*. But, while P deſcribes *q*D, the right line *pm* deſcribes a ſpace equal to the trapezium *qn*ED, by the ſuppoſition; and, its motion being perpetually accelerated during this time, it follows from ax. 2. that the trapezium *qn*ED is leſs than *ex* the ſpace which would be deſcribed in an equal time by *pm* with the motion continued uniformly which it acquires at the term when *p* comes to *d*. And, the trapezium *qn*ED being greater than the parallelogram *rq*, (ſince D*r* is leſs than DE in the ſame proportion as *ek* is ſuppoſed leſs than EG,) it follows, that the parallelogram *ex* is greater than the parallelogram *rq*. But *ex* was found equal to *rq*: And theſe being contradictory, it follows, that the parallelogram *ek* is not leſs than the parallelogram EG. Nor is *ek* greater than EG; and, conſequently, theſe parallelograms are equal to each other. Therefore, when P and *p* come to D and *d*, if the motion of *pm* was continued uniformly from that term, it would deſcribe a ſpace equal to the parallelogram EG, in the ſame time that the point P deſcribes DG with its uniform motion: and, the fluxion of the baſe AD being

being represented by DG, the fluxion of the triangle ADE (which is measured by the velocity of *pm* at the term when P comes to D and *p* to *d*) is represented by the parallelogram EG.

87. *Case* 2. Let the base decrease uniformly, or the motion of P be uniform from O towards A; and the motion of the right line *pm* from *o* towards *a* shall be perpetually retarded, by lemma 3. In this case, (the construction and figure being the same as in the former,) the space *ek* is greater than the parallelogram *eb*, or the trapezium DECB, by ax. 3. and *ek* is less than the parallelogram *hd*, or the trapezium GHED, by ax. 4. I say further, that the parallelogram *ek* is precisely equal to the parallelogram EG. For, if *ek* be not equal to EG, let it first be greater than EG; and, DR being supposed to be greater than DE in the same proportion, let RL parallel to the base meet GH in L: and *ek* will be equal to RG. Then, in the same manner as in the 85th article, the parallelogram RG, or *ek*, is to the parallelogram RQ as the base DG is to DQ: and it follows from theor. 1. that, in the time P describes QD, *pm* would describe a space equal to the parallelogram RQ by its motion continued uniformly from the term when P comes to D; and, by ax. 4. this space must be less than the trapezium QNED, which is equal to the space that was described by *pm* with its retarded motion before *p* came to *d* while P described QD. But the parallelogram RQ is greater than the trapezium QNED, since DR is greater than DE in the same proportion as the parallelogram *ek* is supposed greater than EG: And these being contradictory, it follows, that the parallelogram *ek* is not greater than the parallelogram EG. Let *ek* therefore be supposed less than EG, and D*r* less than DE in the same proportion; then, completing the parallelogram *r*G, as in the last article, *r*G shall be equal to *ek*: and, since the parallelogram *r*G is to *rq* as the base DG is to D*q*, it follows from theor. 1. that, in the same time P describes D*q*, the right line *pm* would describe a space equal to the parallelogram *rq*, by its motion continued uniformly from the term when *p* comes to *d*. But this space (by ax. 3.) must be greater than the trapezium DE*nq*, which is equal to the space described in the same time by *pm* when its motion is perpetually retarded from the same

ſame term: and the parallelogram *rq* is leſs than the trapezium DE*nq*, ſince D*r* is leſs than DE in the ſame proportion as the parallelogram *ek* is ſuppoſed to be leſs than EG. But theſe are contradictory; and therefore *ek* is not leſs than EG. Nor is *ek* greater than EG; and therefore *ek* and EG are equal.

FIG. 19. 88. *Caſe* 3. Let the triangle APM increaſe uniformly, or the motion of *pm* from *a* towards *o* be uniform; and the motion of P from A towards O ſhall be perpetually retarded, by art. 84. In this caſe, if *ek* be ſuppoſed greater than EG, let DR be greater than DE in the ſame proportion; and the parallelogram RG ſhall be equal to *ek*. Let RL parallel to the baſe meet EH in N, and NQ parallel to DE meet the baſe in Q. Then, ſince the parallelogram RG is to RQ as DG is to DQ; and it is ſuppoſed, that, in the time *pm* deſcribes the parallelogram *ek* (or RG) with its uniform motion, the point P would deſcribe DG with its motion continued uniformly from the term when it comes to D: it follows, from theor. 1. that the point P would deſcribe DQ, by the ſame motion continued uniformly, in the time *pm* deſcribes a ſpace equal to RQ. Therefore the point P would deſcribe a line leſs than DQ, by the ſame motion continued uniformly, in the time *pm* deſcribes with its uniform motion a ſpace equal to the trapezium DENQ, which is leſs than RQ. But, while *pm* deſcribes a ſpace equal to DENQ, the point P deſcribes DQ with a motion perpetually retarded, (by art. 84.) and therefore, by ax. 3. it would deſcribe a line greater than DQ in this time, by its motion continued uniformly from the term when it comes to D: And theſe being contradictory, it appears that *ek* is not greater than EG. Let *ek* be now leſs than EG; and, if D*r* be leſs than DE in the ſame proportion, *ek* ſhall be equal to *r*G. Let *rl* parallel to the baſe meet CH in *n*, and *nq* parallel to DE meet the baſe in *q*. Then, ſince the parallelogram *r*G is to *rq* as the baſe DG is to D*q*, it follows, from theor. 1. that, in the ſame time *pm* deſcribes a ſpace equal to *rq* with its uniform motion, the point P would deſcribe a line equal to D*q* by its motion continued uniformly from the term when it comes to D; and therefore P would deſcribe a greater line than D*q*, by the ſame motion continued uniformly, in the time *pm* deſcribes a ſpace equal to the trape-

trapezium *qn*ED, which is greater than the parallelogram *rq*. But in this time the point P describes *q*D with a retarded motion, (by art. 84.) and therefore it would describe a less line than *q*D in the same time with the motion that remains when it comes to D continued uniformly, by ax. 4. And these being contradictory, it follows, that the parallelogram *ek* is not less than EG. Nor is *ek* greater than EG; and therefore these parallelograms are equal to each other.

89. *Case* 4. Let the triangle APM decrease uniformly, or the motion of *pm* from *o* towards *a* be uniform; and the motion of P from O towards A shall be perpetually accelerated, by art 84. In this case, if *ek* was equal to any parallelogram RG greater than EG, (the construction being similar to that of the last article,) the point P would describe a line equal to DQ, by its motion continued uniformly from the term when it comes to D, in the same time *pm* with its uniform motion describes a space equal to the parallelogram RQ; and the point P would describe a line less than DQ, by the same motion continued uniformly, in the time *pm* describes a space equal to the trapezium QNED. But, while *pm* describes a space equal to this trapezium, the point P describes QD with an accelerated motion; and it would describe a line greater than QD in this time, with the motion it acquires when it comes to D continued uniformly, by ax. 1. And these being contradictory, it follows that *ek* is not greater than EG. If *ek* was equal to any parallelogram *r*G less than EG, then the point P would describe D*q*, by its motion continued uniformly from the term when it comes to D, in the same time *pm* describes a space equal to the parallelogram *rq*; and P would describe a greater line than D*q*, by the same motion continued uniformly, in the time *pm* describes a space equal to the trapezium DE*nq*, which is greater than the parallelogram *rq*. But, while *pm* describes a space equal to DE*nq* by its uniform motion, the point *p* describes the line D*q* with an accelerated motion; and it would describe in this time a less line than D*q*, with its motion continued uniformly from the term when it comes to D, by ax. 1. And these being contradictory, it appears, that the parallelogram *ek* is not less than EG. But it is not greater than EG; and therefore

fore these parallelograms are equal to each other, when either the base AD, or the triangle ADE, increase or decrease uniformly.

90. The last three cases might have been demonstrated from the first, by art. 47. & 57. But, for the illustration of this method, we chose to deduce them immediately from the axioms and the first theorem. When the motion of *pm* is accelerated, and the motion of P accelerated or retarded, the proposition may be demonstrated in the same manner. But all the other cases of this proposition are briefly deduced from the first, by the eleventh theorem, thus. In general, let the motion of P, while it describes AO, be accelerated or retarded at pleasure; but let it come to D with a motion, that, continued uniformly for any given time, would generate the line DG: and the right line *pm* shall come to *de* with a motion which, if it was continued uniformly for the same time, would generate a parallelogram *ek* equal to the parallelogram EG. For, suppose QN parallel to PM to generate the triangle AQN; let the motion of Q be uniform, and equal to that with which P comes to D; let *qn*, equal and parallel to *pm*, generate the parallelogram *aqnf*, always equal to the triangle AQN: and, if the motion of *qn* was continued uniformly from the term when *q* comes to *d*, it would describe a space equal to the parallelogram EG in the same time Q describes DG, by the first case, (art. 85. & 86.) Therefore, since the velocity of Q is constant, the velocity of *qn* increases or decreases as DE, or as AD increases or decreases; and, consequently, it is accelerated or retarded in a continued manner. It is evident, that *aq* is determined from AQ in the same manner as *ap* is determined from AP. Therefore, since the points P and Q are supposed to come to D with equal velocities, the points *p* and *q* shall come to *d* with equal velocities, by theor. 11. and the velocity of *pm* at that term is equal to the velocity of *qn*. From which it follows, that if the motion of *pm* was continued uniformly from the term when *p* comes to *d*, it would describe a space equal to the parallelogram EG, in the same time that the point P would describe DG with its motion continued uniformly from the same term. Therefore, in general, the motion with which the base AD flows would generate DG, and the motion with which the triangle ADE flows, continued uniformly,

FIG. 20.

uniformly, would generate a ſpace equal to the parallelogram EG, in the ſame time: ſo that the fluxion of the baſe AD is repreſented by DG, and at the ſame time the fluxion of the triangle ADE by the parallelogram EG.

91. COROLLARY I. The motion with which the triangle flows is the ſame, whether it increaſe or decreaſe with an uniform or with a variable motion, when the baſe is of the ſame magnitude, and flows with the ſame motion. For, when AD and DG are given, the parallelogram EG is of a given magnitude.

92. COR. II. The triangle ADE, any trapezium ADEF when the line FE is given in poſition, and the parallelogram ADEL when the ſide AL is invariable, flow with the ſame motion when the motion with which their baſe AD flows is given. For all thoſe motions continued uniformly would generate the ſame parallelogram EG in the ſame time. FIG. 21.

93. COR. III. While the baſe AD by increaſing uniformly acquires the augment DG, the triangle ADE acquires the augment DGHE. But it is only the part EG of this augment that can be ſaid to be generated by the motion with which the triangle ADE flows at the term when P comes to D. The part EIH is generated in conſequence of the accelerations of this motion: for the ſpace EG is all that would be generated by the motion with which the triangle ADE flows, if that motion was continued uniformly, without any further acceleration, for the time in which the baſe AD acquires the augment DG. If the motion of *pm* was to be accelerated no more after it arrives at *d*, then we have ſhewn that *pm* proceeding with an uniform motion would deſcribe a ſpace equal to EG, and not to the trapezium DEHG, in the time P deſcribes DG; and, in meaſuring this motion of *pm*, or the fluxion of the triangle ADE, the part EIH of the increment which the triangle acquires in this time ought to be rejected. When the baſe decreaſes uniformly, the triangle decreaſes with a retarded motion, the parallelogram EB, or EG, is equal to the ſpace that would be generated by this motion, or the motion of *pm* at *d*, if it was continued uniformly, while P deſcribes DB: but a leſs ſpace DECB is generated in this time by the retarded motion with which the triangle flows; and the difference ECS ariſes from

the retardation of that motion. In general, whether the motion of P be uniform or variable, whatever is generated by the motion with which the triangle flows, more than the parallelogram EG, (in the time P would describe DG by its motion at D,) arises from the acceleration of this motion; and whatever is generated less than EG is owing to its retardation. Those accelerations or retardations may observe various laws. They depend upon the motion of the point P before or after it comes to the term D; and are different when the point E describes different right lines. But the motion with which the triangle ADE flows when P comes to D, is not affected by them, and depends upon the motion of P at the term when it comes to D and the magnitude of the right line DE only, the angle EDG being given.

FIG. 18. 94. COR. IV. When the base increases uniformly, the triangle increases with a motion that is uniformly accelerated. For, if DG be described by P with any uniform motion in a given time, DG will be of an invariable magnitude, and the parallelogram EG will increase in the same proportion as DE or AD, and therefore will increase uniformly; so that the velocity of *pm*, or of the point *p*, will be as the time from the beginning of the motion, supposing the point P to begin its motion at A. When the base decreases uniformly, the motion with which the triangle flows is uniformly retarded: for it decreases in the same proportion as DE or AD decreases; and the velocity of *pm*, or of the point *p*, decreases in the same proportion as the time that remains to flow till P come to A. The motion of *pm*, or of *p*, in the first case, is similar to that of heavy bodies descending by the action or influence of an uniform gravity; and the motion of *pm*, if it was to be accelerated no more after it comes to *d*, would be similar to the motion of a heavy body, if, after a like term, the action of gravity upon it was to cease, or was destroyed by an equal opposite action, pressure or resistance. And thus it appears, that we have made no suppositions, in demonstrating this proposition, but such as are not only easily conceived (which however is all that is required in geometry) and generally admitted, but are also founded in nature and common observation, and are illustrated by the motions that are most universally known. When the base AD decreases

decreaſes uniformly, the motion of *pm*, or of the point *p*, is ſimilar to the motion of a heavy body riſing in a line perpendicular to the horizon againſt the action of an uniform gravity.

95. Cor. V. It may be worth while to obſerve, that the doctrine of motions that are accelerated or retarded uniformly, is eaſily demonſtrated from the two firſt caſes of this propoſition, without ſuppoſing any quantities indiviſible or infinitely diminiſhed. When the motion of P is uniform, AP repreſents the time; the ſpace deſcribed by *pm*, being always equal to the triangle APM, increaſes in the duplicate ratio of the time AP, (Elem. 19. 6.) and the ſpace deſcribed by the point *p* increaſes in the ſame proportion. The velocity of *p* increaſes uniformly, in the ſame proportion as AP the time from the beginning of the motion, by the laſt corollary. From which it follows, converſely, that if the motion of any point, as p, be accelerated uniformly, or in the ſame proportion as AP the time from the beginning of the motion, the ſpace deſcribed by it from the beginning of the motion ſhall increaſe in the duplicate ratio of this time. For, if the velocity of p be equal to the velocity of *p* at any term of the time, their velocities, and conſequently the ſpaces deſcribed by them in the ſame time, (by theor. 4.) muſt be always equal. And this is the celebrated theorem diſcovered by Galileus. Becauſe the parallelogram EG is to the parallelogram AE as the baſe DG is to AD; it appears, that, in the time P deſcribes AD, the right line *pm* would deſcribe a ſpace equal to the parallelogram AE, if the motion it acquires at *d* was continued uniformly. But the parallelogram AE is double of the triangle ADE, or of the parallelogram *ae*. Therefore the point *p* would deſcribe a line double of *ad*, if the motion it acquires at *d* was continued uniformly, in the ſame time that it deſcribes *ad* with a motion that is accelerated uniformly: which is another of his theorems. Becauſe the trapezium BCHG is double of the parallelogram EG, it follows, that the ſpace deſcribed by a motion that is uniformly accelerated, is equal to the ſpace that would be deſcribed in an equal time by the motion at the middle term of that time continued uniformly; as was ſhewn in a different manner in art. 72. When the baſe AP decreaſes uniformly, the velocity of *p* from *o* towards *a* decreaſes Fig. 21.

uniformly, and may be always reprefented by AP. The fpace *pa* that remains to be defcribed by the point *p* before its motion be at an end, decreafes in the fame ratio as the triangle APM decreafes, or in the duplicate ratio of AP; that is, in the duplicate ratio of the velocity of *p*, or of the time that remains to flow till the end of the motion. Therefore the fpaces that may be defcribed by bodies before their motions be deftroyed, when they are continually retarded by an uniform gravity, (or by any uniform power or refiftance that diminifhes their velocities equally in any equal times,) are in the duplicate ratio of their velocities.

96. Cor. VI. If the angle APM be a right one, and PM be double of AP, the triangle APM fhall be always equal to a fquare defcribed upon AP; and the motions with which this fquare and the triangle APM flow fhall be always equal, by theor. 3. Therefore, when P comes to D, the motion with which a fquare upon AD flows, continued uniformly, would generate a rectangle equal to EG, in the time P defcribes DG. But the rectangle EG is equal, in this cafe, to a rectangle contained by 2AD and DG; and this rectangle reprefents the fluxion of the fquare, when DG reprefents the fluxion of AD the fide of the fquare. The parallelogram *ad* being equal to the fquare of AD, the right lines *af*, AD and *ad* are in continued proportion; and, the rectangles *ek* and EG being equal, DG (which reprefents the fluxion of AD) is to *dk* (which reprefents the fluxion of *ad*) as *af* is to DE, or 2AD; and DG is to one half of *dk* in the fubduplicate ratio of *af* to *ad*. Therefore, when three quantities are in continued proportion, and the firft is invariable, the fluxion of the fecond term is to the fluxion of the third, as the firft term is to twice the fecond term; and the fluxion of the fecond term is to one half of the fluxion of the third term in a fubduplicate ratio of the firft term to the third.

97. Cor. VII. When the bafe flows uniformly, the velocity of the motion with which the triangle flows increafes in the fame proportion as PM or AP increafes, and is equally augmented in any equal times. Therefore the fluxion of this velocity, or the fecond fluxion of the triangle, is conftant: and if we conceive the continual acceleration of this velocity to be the

effect

effect of ſome power, the exertion of this power muſt be ſuppoſed uniform. When P comes to D, the motion with which the triangle flows, continued uniformly, would generate the parallelogram EG, in the ſame time P deſcribes DG. When P comes to G, the motion with which the triangle then flows, continued uniformly, would generate the parallelogram DH in the ſame time. The difference of theſe parallelograms, or the parallelogram IN, repreſents the ſecond fluxion of the triangle ADE; and IN is double of the triangle EIH, which is the part of the increment of the triangle that is generated in conſequence of the uniform acceleration of the motion with which it flows while P deſcribes DG. It is evident, that, if the motion of P be increaſed or diminiſhed, then the motion with which the triangle ADE flows, or its fluxion, is increaſed in the ſame proportion; becauſe the parallelogram EG increaſes in the ſame proportion as its baſe DG. But the triangle EIH, or the parallelogram IN which meaſures the ſecond fluxion of the triangle, increaſes or decreaſes in the duplicate ratio of DG, or of the ſpace EG which meaſures its firſt fluxion. And, the velocity of P remaining, if the time in which DG is ſuppoſed to be deſcribed be continually diminiſhed, then EG which meaſures the firſt fluxion of the triangle decreaſes in the ſame proportion; but IN is diminiſhed in the duplicate ratio of this time, and its ratio to EG may become leſs than any aſſignable ratio. The parallelogram IN (which meaſures the ſecond fluxion of the triangle ADE) is equal to the difference of the increments DECB, DGHE which are generated while P deſcribes BD and DG. The increment DGHE is equal to the parallelogram EG (which meaſures the firſt fluxion of the triangle) added to one half of IN which meaſures its ſecond fluxion. Theſe things agree with the 74th, 75th, 76th and 77th articles, and illuſtrate them. When the baſe flows uniformly, it is evident that the triangle has no third fluxion: and what we have ſaid of the triangle APM is eaſily applied to the right line *ap*, which always flows in the ſame manner, becauſe the rectangle contained by it and the invariable right line *af* is always equal to the triangle, by the ſuppoſition.

LEMMA

LEMMA IV.

98. *Let the points* P *and* Q *describe the right lines* AO, AV *given in position, with motions that are either uniform, or are always to each other in an invariable ratio; and, the parallelogram* APRQ *being completed, the point* R *shall be always found in the same right line.*

Fig. 22. & 23.

While the point P describes BD and DG upon the line AO, let Q describe KL and LM upon the line AV; complete the parallelograms ABCK, ADEL and AGHM; let CN parallel to AO meet DE and GH in S and N; and let EI parallel to AO meet GH in I. Because the velocities of P and Q are to each other in an invariable ratio, it follows from theor. 6. that BD is to DG as KL is to LM. Therefore BD is to BG as KL is to KM, or CS is to SE as CN is to NH; and, consequently, any three places C, E and H of the point R being in a right line, it follows, that the point R is always found in the same right line.

PROP. III.

99. *The fluxions of the right lines* AD *and* AL *being represented by* DG *and* LM, *the fluxion of the rectangle* AE, *contained by* AD *and* AL, *is measured accurately by the sum of the rectangles* EG *and* EM *when these lines increase or decrease together, but by the difference of* EG *and* EM *when one of those lines decreases while the other increases.*

Fig. 22. & 23.

Fig. 22. *Case* 1. Let the sides AP, AQ of the rectangle AR both increase, or both decrease, together, by the uniform motions of the points P and Q. When P comes to D, let Q come to L, and R to E; and, while P describes DG, let Q describe LM, and R describe EH: and the point R shall be always found in the right

right line EH, by the last lemma. Produce HE, and let it meet AD in F before it meets with AL: then, the rectangle AR being always equal to the sum of the triangle FPR and trapezium AFRQ, the increment or decrement of the rectangle is always the sum of the increments or decrements of the triangle and trapezium that are generated in the same time; and, by theor. 7. the motion with which the rectangle flows, is always equal to the sum of the motions with which that triangle and trapezium flow. But, by prop. 2. (art. 85.) if the motions with which the triangle FDE and trapezium AFEL flow were continued uniformly, the parallelograms EG and EM would be generated by them in the same time that the points P and Q describe DG and LM with their uniform motions. Therefore, if the motion with which the rectangle AE flows was continued uniformly, a space equal to the sum of the parallelograms EG and EM would be generated by it in the same time; and, if the fluxions of the sides AD and AL be represented by DG and LM, the fluxion of the rectangle AE must be represented by the sum of the rectangles EG and EM. The demonstration is the same when HE produced meets AL before it meets with AD: only, in this case, EM represents the fluxion of the triangle, and EG the fluxion of the trapezium; and, if HE produced pass through A, the point F coincides with A, and the rectangle AE is the sum of two triangles the fluxions of which are represented by the rectangles EG and EM.

100. *Case* 2. Let the side AQ decrease while the side AP increases; and, the motions of P and Q being still uniform, the point R shall describe a right line EH, by the last lemma. Because the angle MHE is acute, and the angle HML right, HE produced shall meet with AL produced beyond L in some point *f*; and it shall meet with AD produced beyond D in some point F. The rectangle AE in this case is always the excess of the trapezium ADE*f* above the triangle *f*LE; and the motion with which it flows is always equal to the difference of the motions with which that trapezium and triangle flow, by art. 41. Therefore, if the fluxion of the sides AD and AL be represented by DG and LM, the fluxion of the rectangle AE shall be represented by the difference of the rectangles EG and EM. FIG. 23.

101. *Case*

101. *Case* 3. When the motions of the points P and Q are variable, but are in an invariable ratio to each other, the point R ſtill deſcribes a right line, by the laſt lemma; and the motion with which the rectangle AE flows, is ſtill meaſured by the ſum or difference of the rectangles EG and EM when the motions of P and Q, at the term when they come to D and L, are meaſured by DG and LM, by prop. 2. In general, let FIG. 24. the motions of P and Q be varied at pleaſure; and let them come to D and L with motions by which continued uniformly they would deſcribe DG and LM in any given time. Let the right line AC conſtitute with AD an angle that is half a right one; produce ED and EL till they meet AC in C and K: and KE, or CE, ſhall be always equal to the ſum of AD and AL. Upon AD produced let AF be always equal to CE, and FR parallel to DE meet AC in R; and the triangle AFR (which is equal to KEC) ſhall always exceed the rectangle AE by the two triangles ADC, ALK. Therefore the motion with which the rectangle AE increaſes, is equal to the exceſs of the motion with which the triangle AFR increaſes above the ſum of the motions with which the triangles ADC, ALK increaſe. When the ſides AD, AL increaſe together, take FT equal to the ſum of DG and LM, but equal to their difference when AL decreaſes while AD increaſes. Let GQ and TX, parallel to DE, meet AC in the points Q and X; let MY, parallel to AD, meet AC in Y; let CP and RZ, parallel to AD, meet GQ and TX in P and Z; and let KS, parallel to AL, meet AC in Y. By prop. 2. if the motion with which the triangles AFR, ADC, ALK increaſe were continued uniformly, they would generate the ſpaces FZ, DP, LS in the ſame time that the motions with which their baſes increaſe, continued uniformly, would generate the right lines FT, DG, LM. But, when the ſides AD, AL increaſe together, FT is equal to the ſum of DG and LM; and FR is equal to KE, or EC: and therefore FZ is equal to the ſum of the rectangles ES and EP. From which ſum deduce DP and LS, which meaſure the motions with which the triangles ADC, ALK increaſe, and the remainder is the ſum of EG and EM; which therefore meaſures the motion with which the rectangle AE flows. But if AL decreaſe while AD increaſes,

ſes, then, ET being equal to the difference of DG and LM, the rectangle FZ will be equal to the difference of the rectangles EP and ES; and, becauſe the triangle ALK decreaſes, the motion with which the ſum of the triangles ADC, ALK increaſes will be meaſured by the difference betwixt DP and LS. From FZ, or the difference betwixt EP and ES, ſubduct the difference betwixt DP and LS, and the remainder will be equal to the difference betwixt EG and EM; which therefore meaſures the motion with which the rectangle AE flows when DG and LM meaſure the motions with which its ſides AD and AL flow. If DG be equal to LM, the parallelogram EP is equal to ES; the line FT, and the motion with which the triangle AFR flows, vaniſh; the motion with which the rectangle AE increaſes is equal to the motion with which the ſum of the triangles ADC, ALK decreaſes, and therefore is meaſured by the difference betwixt LS and DP; which is equal to the difference betwixt EG and EM, becauſe ES the ſum of EM and LS is equal to EP the ſum of EG and DP. If LM be greater than DG, then the triangle KEC (or AFR) decreaſes, and FT muſt be taken from F towards A: but the motion with which AE increaſes will ſtill be meaſured by the exceſs of EG above EM. When EG is equal to EM, then this motion vaniſhes; and, when EG is leſs than EM, the rectangle AE decreaſes by a motion that is ſtill meaſured by their difference. Thus it appears, that, whether the ſides AD and AL flow with uniform or variable motions, and the point E deſcribe a right line or not, when the fluxions of the ſides AD and AL are meaſured by DG and LM, the fluxion of the rectangle AE is meaſured by the ſum or difference of the rectangles EG and EM, their ſum when the ſides increaſe or decreaſe together, their difference when one ſide decreaſes while the other increaſes. We ſhall demonſtrate this propoſition in a different manner afterwards. Fig. 25.

102. Cor. When the ſides of the rectangle AE increaſe with uniform motions, the rectangle flows with a motion that is uniformly accelerated: For the rectangle is always equal to the ſum of the triangle FDE and trapezium AFEL; both of which increaſe with motions that are uniformly accelerated, by cor. 4. prop. 2. While the ſides AD and AL acquire the augments DG Fig. 22.

and LM, the rectangle AE is increased by the space DELMHG: but there is no more than the parts EG and EM of this increment generated in consequence of the motion with which the rectangle AE flows at the term or moment when P comes to D and Q to L. The part I*b* is generated in consequence of the uniform acceleration of this motion while the points P and Q describe DG and LM. This part is rejected in measuring the motion with which the rectangle AE flows at that term; but the increase or acceleration of this motion generated in a given time, or the uniform power by which this increase may be conceived to be produced, may be measured by it. A space double of I*b* would be generated in the same given time by an uniform motion equal to the difference of the motions with which the rectangles AH and AE flow; and therefore the second fluxion of the rectangle AE may be measured by that space; as was demonstrated in the fifteenth theorem.

103. In the second case, the fluxion of the rectangle AE may be expressed by the rectangle that is contained by LM and the difference of FD and AD, or of AF and 2AD; for that rectangle is equal to the difference of the rectangles EG and EM. In this case, the fluxion of the rectangle AE continually decreases while AD increases; it vanishes when AD is equal to DF or to one half of AF; and the rectangle AE is greatest at that term. When AD becomes greater than FD, the fluxion of the rectangle AE becomes negative; that is, the rectangle decreases continually till it vanish when AD becomes equal to AF. The second fluxion of the rectangle is invariable, and is measured by 2I*b*, as in the first case: but it is considered as negative, because it continually diminishes the first fluxion, or retards the motion with which the rectangle increases till AD become equal to one half of AF, and accelerates the motion with which the rectangle decreases after that term. If we suppose,
FIG. 22. in the first case, the points P and Q to move from O and V towards A, and their motions to be continued after P comes to F and Q to A, we shall have the first and second cases comprehended in one view.

104. We have insisted so much on the two preceeding propositions for the reasons mentioned in the 69th and 85th articles.

ticles. In delivering those elements, we have endeavoured to avoid every thing that might appear abstruse or obscure, and to obviate the objections that have been advanced lately against this doctrine, though we have not taken particular notice of them. In the third lemma, we demonstrated that the motion with which the triangle flows is perpetually accelerated while the base flows uniformly. Nor do we see how this can be disputed, since that motion cannot be supposed to be either uniform or retarded for any part of the time how small soever. Some Philosophers may be of opinion, that a line or velocity cannot be conceived to increase or decrease continually, but by certain small indivisible increments or decrements only. How repugnant soever this principle may be to what is demonstrated by Geometricians, the preceeding propositions and a great part of this doctrine may be easily adapted to it, as we hinted in the 7th article. It was in effect in this form that some part of its elements first appeared in the method of indivisibles: but we proceed to establish it on the more accurate principles of the ancients.

CHAP. III.

Of the Fluxions of plane curvilineal Figures.

LEMMA V.

105. *The base of any figure being supposed to increase uniformly, the area flows with an accelerated or retarded motion, according as the ordinates increase or decrease while the base increases.*

While the point P describes the base by moving from A towards *a*, or from *a* towards A; let the right line PM, by moving parallel to itself from AF towards *af*, or from *af* towards AF, generate the area APMF or *a*PM*f*. Let *q*D and DQ be e- Fig. 26.

 qual

qual parts of the base described by P in any equal times; let *qn*, DE and QN, parallel to PM, meet the curve, or any continued line FE*f*, in *n*, E and N: Then it is manifest, that, when the ordinates from *qn* to QN increase, the area DENQ is greater than the area DE*nq*. Therefore, when AP increases uniformly, or the point P describes *q*D before it describes QD, and the right line PM generates the area *q*DE*n* before it generates DQNE, the motion with which the area flows is always accelerated: but if *a*P increase uniformly, or the point P move from Q to *q*, the increments of the area *a*PM*f* which are generated in any equal times that succeed after one another, perpetually decrease, and the motion with which it flows is always retarded.

106. It is evident, in the same manner, that, when the ordinates decrease while the base AP decreases, the area APMF decreases with a retarded motion: but, when the ordinates increase while the base *a*P decreases, the area *a*PM*f* decreases with
FIG. 27. an accelerated motion. It is also manifest, that, when the area APMF is supposed to flow with an uniform motion, so that any increments or decrements *qn*ED, DENQ generated in equal times are always equal, then *q*D is greater than DQ; and the base AP increases with a retarded motion, or decreases with an accelerated motion: but the base *a*P increases with an accelerated motion, or decreases with a retarded motion. It is easy to represent the motion with which the area flows by the motion of a given right line *pm*, that, by moving parallel to itself, may be supposed to generate a parallelogram always equal to the area, as in the preceeding chapter. We omit this right line *pm* in the following propositions; but it may be easily supplied by the reader if he pleases.

PROP. IV.

107. *The fluxion of the base* AD *being represented by* DG, *the fluxion of the area* ADEF *is accurately measured by the parallelogram* EG.

FIG. 26. First, let the base AP increase uniformly, and the ordinates PM increase while the base increases. The area flows, in this case,

caſe, with an accelerated motion, by the laſt lemma; and, if this motion was continued uniformly from the term or moment when P comes to D, a ſpace would be generated by it greater than BDEC, (by ax. 2.) but leſs than DGHE, (by ax. 1.) in the ſame time P deſcribes DG with its uniform motion. Let this ſpace be ſuppoſed equal to X, and X ſhall be equal to the parallelogram EG. For, if it be ſaid to be greater than EG, let DR be greater than DE in the ſame ratio; complete the parallelogram DRLG, and it ſhall be equal to X, becauſe it is to EG as DR is to DE, (Elem. 1. 6.) or as X is to EG. Let RL meet the curve CH in N, and NQ parallel to DE meet the baſe in Q. Then, ſince the parallelogram RG is to RQ as the baſe DG is to DQ, it follows from the firſt general theorem, (art. 16.) that a ſpace equal to the parallelogram RQ would be generated by the motion with which the area APMF flows, continued uniformly from the term or moment when P comes to D, in the ſame time P deſcribes DQ. But, while P deſcribes the line DQ, the area DENQ is generated by the accelerated motion with which the area flows; and DENQ muſt be greater than the ſpace which would have been generated in this time if the motion with which the area flows had been continued uniformly from the beginning of the time without any acceleration, by the firſt axiom. Therefore the area DENQ is greater than the parallelogram RQ. But, becauſe DR is greater than DE, (in the ſame proportion as X is ſuppoſed to be greater than EG,) and the ordinates from DE to QN continually increaſe, the parallelogram RQ is greater than DENQ. And theſe being contradictory, it follows that the ſpace X is not greater than EG. If X be ſaid to be leſs than EG, let D*r* be leſs than DE in the ſame proportion; complete the parallelogram D*rl*G, and it ſhall be equal to X. Let *rl* produced meet the curve in *n*, and *nq* parallel to DE meet the baſe in *q*. Then, ſince the parallelogram *r*G is to *rq* as the baſe DG is to D*q*, it follows, that a ſpace equal to the parallelogram *rq* would be generated by the motion with which the area flows, continued uniformly from the term when P comes to D, in the ſame time P deſcribes a line equal to D*q* with its uniform motion. But, while P deſcribed *q*D before it came to D, the ſpace *qn*ED was generated by

by the accelerated motion with which the area flows; and this ſpace *qn*ED muſt be leſs than the ſpace which would be generated by the motion with which the area flows if it was continued uniformly from the term when P comes to D, by the ſecond axiom; that is, *qn*ED is leſs than the parallelogram *rq*: which is abſurd. Therefore the ſpace X is neither greater nor leſs than the parallelogram EG, but preciſely equal to it. When the baſe decreaſes or increaſes uniformly, and the ordinates decreaſe at the ſame time, the area flows with a retarded motion; and the demonſtration is the ſame as that of the 87th article.

Fig. 27. 108. Let the area APMF flow uniformly, and acquire the augment DESK in any given time. Let DG be the ſpace that would be deſcribed by P with its motion at D continued uniformly for the ſame time; and the area DESK ſhall be equal to the parallelogram EG. For, if the ordinates increaſe while the baſe increaſes, the motion of P is retarded, (by art. 106.) and, if the area DESK be ſuppoſed equal to any parallelogram DRLG greater than EG, let RL meet the curve EH in N, and NQ parallel to ED meet the baſe in Q. Then, becauſe the parallelogram RG is to RQ as the baſe DG is to DQ, it follows, that the point P, with its motion continued uniformly from D, would deſcribe the line DQ, in the ſame time that the area by flowing uniformly acquires an augment equal to the parallelogram RQ; and the point P would deſcribe a leſs line than DQ, by the ſame uniform motion, while the area by flowing uniformly acquires the augment DENQ. But, becauſe the motion of P is retarded while the area increaſes uniformly, it follows from the third axiom, that the point P would deſcribe a greater line than DQ, by its motion continued uniformly from D, while the area acquires the augment DENQ. And theſe being contradictory, the area DESK is not greater than EG. In like manner, it appears from the fourth axiom, that the area DESK is not equal to any parallelogram D*rl*G leſs than EG. Therefore the area DESK is equal to the parallelogram EG. When the ordinates decreaſe while the baſe either increaſes or decreaſes, and the area is ſuppoſed to flow uniformly, the baſe flows with an accelerated motion; and the demonſtration is the ſame as in the 89th article. In thoſe caſes, when the motion with which

which the area flows is uniform, the parallelogram EG (which is equal to the ſpace that would be generated by that uniform motion in a given time) is of an invariable magnitude; and the velocity of the point P is always reciprocally as the ordinate PM: whence it may be determined, whether the motion with which the area flows is accelerated or retarded, when the law according to which the velocity of P varies and the nature of the curve FEH are known.

109. In general, whatever the motion of the point P may be, or that with which the baſe flows, the propoſition is demonſtrated univerſally from art. 107. by the ninth and eleventh general theorems in the manner that was deſcribed in the 90th article. Therefore, when the fluxion of the baſe AD is repreſented by DG, the fluxion of the area ADEF is repreſented by the parallelogram EG. And, converſely, when the fluxion of the area is repreſented by the parallelogram EG, the fluxion of the baſe is repreſented by DG.

110. Cor. I. When the fluxion of the baſe AD is given, the fluxion of the area ADEF is the ſame, whatever line be deſcribed by the point M if it paſs through E. If the baſe flow uniformly, whatever is generated more or leſs than the parallelogram EG by the motion with which the area flows, in the time P deſcribes DG, proceeds from the acceleration or retardation of the motion with which the area flows at the term when P comes to D.

111. Cor. II. When the fluxions of the baſes of two figures that are generated in the ſame time are equal, the fluxions of the areas are as their ordinates; and when the ordinates are always in a given ratio, (as in the caſe deſcribed in the introduction, pag. 7. fig. 1.) the areas that are generated in the ſame time are in the ſame given ratio, by the ſixth general theorem. When the fluxions of the baſes are reciprocally as the ordinates, the fluxions of the areas are equal; and when the fluxions of the baſes are always to each other in this ratio, the areas that are generated in the ſame time are equal, by the fourth general theorem.

112. Cor. III. Let AE be a rectangle contained by the two ſides AD, AL; while AD and AL flow with any accelerated Fig. 28.

or

or retarded motions, let E describe any line FE that meets one of the sides AD in F. Then, the rectangle AE being equal to the sum or difference of the areas AFEL and FDE, the fluxion of the rectangle AE shall be measured by the sum or difference of the rectangles EG and EM, when the fluxions of the sides AD, AL are measured by the right lines DG and LM. And thus the third case of the third proposition is briefly demonstrated.

FIG. 29. 113. COR. IV. If the area AD*ef* be always equal to the rectangle contained by the ordinate DE and the right line DG that represents the fluxion of the base AD, then shall the parallelogram *e*G measure the fluxion of the area AD*ef*, or of the parallelogram EG, or the second fluxion of the area ADEF. And, if the area ADef be always equal to the parallelogram *e*G, the parallelogram eG shall measure the third fluxion of the area ADEF. In the same manner the higher fluxions of this area may be represented; but, in some cases, by proceeding thus, we shall come to some fluxion that is constant, which cannot be considered as a flowing or variable quantity.

114. COR. V. From this proposition some general theorems in the doctrine of motion may be easily demonstrated. If the FIG. 26. & 30. base AP represent the time of any motion, PM the velocity at any term of the time P, the space described by the motion in the time AP may be measured by the area APMF; that is, the space described in the time AP shall be to the space described in any other time AD as the area APMF is to the area ADEF. For, the motion of P being uniform, the motion with which the area APMF flows varies in the same proportion as the ordinate PM, by what has been demonstrated; and therefore, if the invariable right line *pm*, by moving parallel to itself, describe a space *am* always equal to the area APMF, the velocity of the right line *pm*, or of the point *p*, shall be always measured by the ordinate PM at any term of the time, as P. From which it follows, conversely, (as in art. 95.) that if we suppose the velocity of the point *p* at any term P to be measured by the ordinate PM, the space described by the point *p* in the time AP shall be measured by the area APMF; or, the space described by *p* in the time AP shall be to the space described by it in the time AD as the area APMF is to the area ADEF. In the

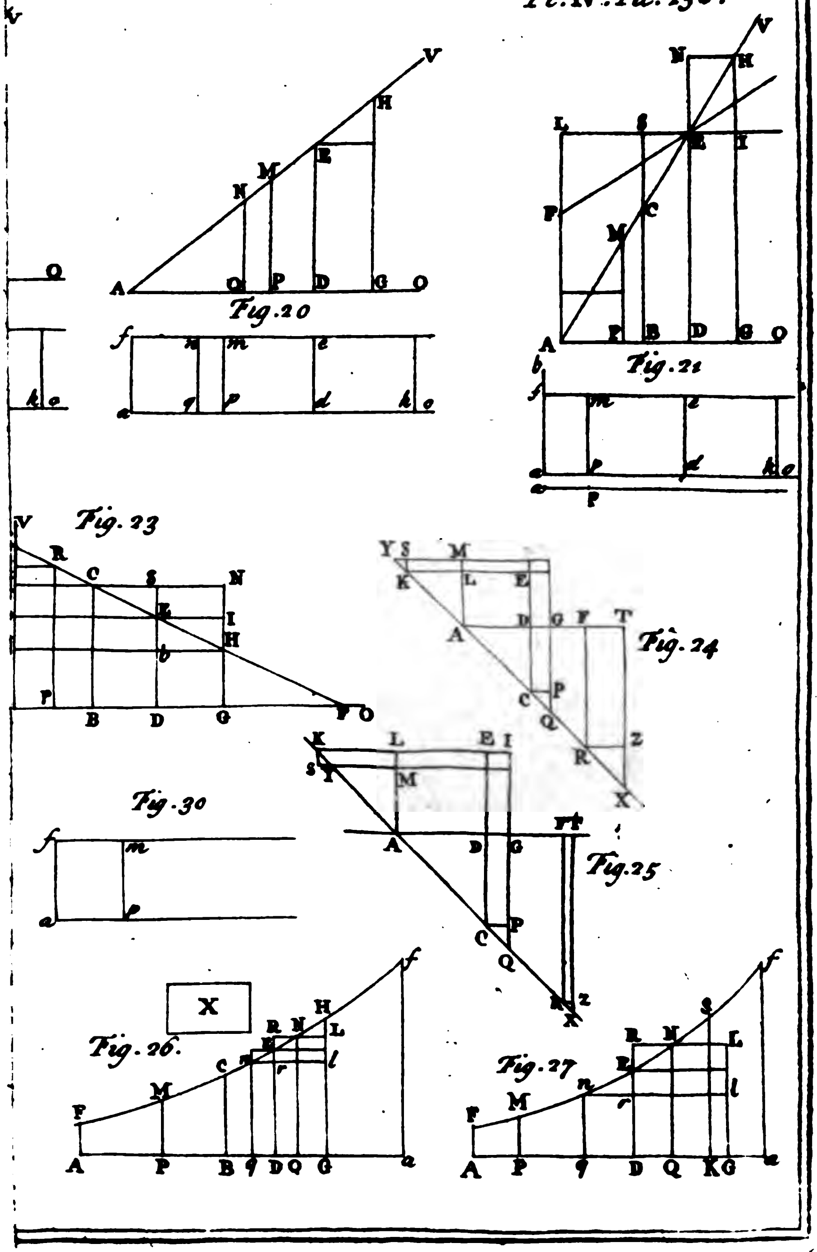


Fig. 20
Fig. 21
Fig. 23
Fig. 24
Fig. 25
Fig. 30
Fig. 26.
Fig. 27
X

the ſame manner, if the action or exertion of any power that accelerates or retards motion be always repreſented by the ordinate PM while the baſe AP repreſents the time, the motion that is generated or deſtroyed by this power in the time AP is to the motion that is generated or deſtroyed in the time AD as the area APMF is to the area ADEF. Quantities of all kinds that increaſe or decreaſe while the time increaſes in any regular manner whatſoever, may be thus ſubjected to menſuration by the quadrature of figures.

115. Cor. VI. There is another general theorem in the doctrine of motion that alſo follows eaſily from this propoſition. Let the baſe AP repreſent the ſpace deſcribed by any motion; Fig. 27. let the ordinate PM be always reciprocally as the velocity; that is, let the velocity at any place P be always to the velocity at any other place D as the ordinate DE is to the ordinate PM; and the time in which the line AP is deſcribed ſhall be to the time in which AD is deſcribed as the area APMF is to the area ADEF. For it follows from this propoſition, (art. 108.) that, if the area APMF flow uniformly, the velocity of the point that deſcribes the baſe ſhall be reciprocally as PM; that is, its velocity at P ſhall be to its velocity at any given term D as DE is to PM. From which it eaſily follows, converſely, (as in art. 95.) that, if the velocity of a point that deſcribes the line AP at any term of that line, as P, be always reciprocally as the ordinate PM, the area APMF ſhall flow uniformly, or in the ſame proportion as the time of the motion.

116. Let a given right line SP, revolving about the center S, generate the circle AD*a*, and meet any curve FEH always in M; then, if the ray SM increaſes while the arch AP increaſes, it is manifeſt, that, when the motion of P in the circumference AD*a* is uniform, the area FSM flows with an accelerated motion. For, if BD and DG be equal arches deſcribed by P in any equal times, and the rays SB, SD, SG meet the curve FMH in C, E and H, the area SEH ſhall be greater than SEC. The ſector ASP and the angle ASP increaſe uniformly in this caſe. If the motion of P be variable, but its velocity at D be meaſured by the arch DG, the motion with which the ſector ASD flows ſhall be meaſured by the ſector DSG, and the

motion with which the angle ASD flows by the angle DSG; as may be demonſtrated in the ſame manner as the 80th article.

PROP. V.

117. *The area* FSE *being ſuppoſed to flow as in the laſt article, let the arch* EI *deſcribed from the center* S *meet the ray* SG *in* I; *and, if the fluxion of the ſector* ASD *be repreſented by the ſector* DSG, *the fluxion of the curvilineal area* FSE *ſhall be accurately meaſured by the ſector* ESI.

FIG. 31.

Let the motion of P in the circumference AD*a* firſt be uniform, and the rays SM terminated by the curve FMH increaſe while AP increaſes, as in the laſt article; and the area FSM ſhall flow with an accelerated motion. Suppoſe that a ſpace equal to X would be generated by this motion continued uniformly from the term when P comes to D, in the ſame time P deſcribes DG; and the ſpace X ſhall be leſs than the area ESH, (by ax. 1.) but greater than the area CSE, (by ax. 2.) If X be not equal to the ſector SEI, let it firſt be greater than that ſector. Produce SE to R till SR be to SE in the ſubduplicate ratio of the ſpace X to SEI; let the arch RL deſcribed from the center S meet the rays SD, SG in R and L, and the curve in N; and the ſpace X ſhall be equal to the ſector SRL. Let SN meet the circle in Q; and, ſince the ſector SRL is to SRN as DG is to DQ, it follows, (by theor. 1.) that, if the motion with which the area FSE flows be continued uniformly, it will generate a ſpace equal to the ſector SRN, in the time that P deſcribes the arch DQ. But this ſpace muſt be leſs than the area SEN which is generated in the ſame time by the accelerated motion with which the area FSM flows, by the firſt axiom. Therefore the ſector SRN is leſs than the area SEN. But this is abſurd; and, conſequently, the ſpace X is not greater than the ſector SEI. If it be ſaid that the ſpace X is leſs than the ſector SEI, let S*r* be leſs than SE in the ſubduplicate ratio of the ſpace X to the ſector SEI. From S as center deſcribe the arch *rl*, meet-

meeting the rays SD, SG in *r* and *l*, and the curve EC in *n*. Let S*n* meet the circumference of the circle AD*a* in *q*; and the ſpace X ſhall be equal to the ſector S*rl*. But S*rl* is to S*rn* as *rl* is to *rn*, or DG to D*q*; and therefore (by theor. 1.) a ſpace equal to the ſector S*rn* would be generated by the motion with which the area FSE flows continued uniformly, during the time in which P deſcribes an arch equal to *q*D; and this ſpace muſt be greater than the area S*n*E which is generated by the accelerated motion with which the area flows while P deſcribes *q*D before it comes to D, by the ſecond axiom; that is, the ſector S*nr* muſt be greater than the area S*n*E: which is abſurd. Therefore the ſpace X is neither greater nor leſs than the ſector SEI, but is preciſely equal to it.

118. The reſt remaining as in the laſt article, ſuppoſe the area FSM to increaſe uniformly, and to acquire the equal increments CSE, ESH in any equal times that immediately ſucceed after each other. Let the rays SC and SH meet the circumference AD*a* in *k* and K; and, ſince the rays from S terminated by the curve CEH are ſuppoſed to increaſe continually from SC to SH, it is manifeſt, that the angle ESH is leſs than ESC, and that the arch DK is leſs than D*k*. Therefore the motion of the point P, in deſcribing the circumference AD*a*, is perpetually retarded. Suppoſe that the point P would deſcribe the arch DG by its motion at D continued uniformly, in the ſame time that the area FSE by flowing uniformly acquires the augment ESH; and the area ESH ſhall be equal to the ſector ESI. For, if ESH be greater than ESI, let SR be greater than SE in the ſubduplicate ratio of ESH to ESI; let the arch RL deſcribed from the center S meet SG in L, and the curve EH in N; and let SN meet the circumference DG in Q. Then, ſince the ſector SRL is to the ſector SRN as the arch RL is to the arch RN, or as DG is to DQ, it follows, that, in the time the area FSE by flowing uniformly acquires an augment equal to the ſector SRN, the point P would deſcribe DQ with its motion continued uniformly from the term when it comes to D; and, in the time that the area FSE acquires the augment ESN, (which is leſs than the ſector SRN,) the point P would deſcribe an arch leſs than DQ by the ſame motion continued uniformly. But, FIG. 32.

while the area FSE acquires the augment ESN, the point P with a retarded motion deſcribes DQ; and it would deſcribe an arch greater than DQ in the ſame time by its motion continued uniformly from D, by the third axiom. And theſe being contradictory, it follows, that the area ESH is not greater than the ſector ESI. In the ſame manner it appears, that ESH is not equal to any ſector S*rl* leſs than SEI.

119. In general, all the other caſes of this propoſition are deduced from the firſt, by the ninth and eleventh theorems, in the ſame manner that we demonſtrated the 90th article: ſo that, when the fluxion of the arch AD is repreſented by DG, or the fluxion of the ſector ASD by the ſector DSG, the fluxion of the area FSE is expreſſed accurately by the ſector ESI; and, converſely, when the fluxion of the area FSE is expreſſed by the ſector ESI; produce SI till it meet the circumference AD*a* in G, and DG ſhall repreſent the fluxion of the arch AD.

120. Cor. I. The fluxion of the ſector ASD is to the fluxion of the area FSE in the duplicate ratio of SD to SE. When the arch AD, or the angle ASD, flows uniformly, the fluxion of the area FSE increaſes or decreaſes in the duplicate ratio of the ray SE. But, when the area FSE flows uniformly, the fluxion of the arch AD, or of the angle ASD, is reciprocally as the ſquare of the ray SE.

Fig. 32. 121. Cor. II. In the 118th article, where the area was ſuppoſed to flow uniformly, it was ſhewn, that, if the ſector ESI be equal to the area ESH, and SI produced to the circumference AD*a* meet it in G, then DG is the arch that would be deſcribed by P with its motion at D continued uniformly while the area FSE acquires the augment ESH. From this it follows, that, if the angular motion of the ray SE was continued uniformly, the angle ESI would be generated by it in the ſame time. Let SH meet the arch EI in R; and the angle ESR, which is actually generated by the ray SP revolving about the center S while the area is increaſed by ESH, is to the angle ESI that would have been generated in the ſame time if the angular motion of SE about S had been continued uniformly, as the ſector ESR is to the area ESH; and the difference of thoſe angles is to the latter angle ESI as the area ERH is to the area ESH.

122. The

122. The similarity of curvilineal figures may be derived from that of rectilineal figures that are always similarly described in them; or, we may comprehend all sorts of similar figures, planes or solids, in this general definition: Figures are similar when they may be supposed to be placed in such a manner, that, any right line being drawn from a determined point to the terms that bound them, the parts of the right line intercepted betwixt that point and those terms are always in one constant ratio to each other. Thus the figures ASD, *a*S*d* are similar, FIG. 33.
when, any line SP being drawn always from the same point S meeting AD in P, and *ad* in *p*, the ratio of SP to S*p* is invariable. It is manifest, that the rectilineal inscribed figures APDS, *apd*S are similar, in this case, according to the definition of such figures that is given in the Elements. If the right lines PM and *pm* parallel to each other meet SA in M and *m*; then PM shall be to *pm* as SM is to S*m*, or as SA is to S*a*; and AM is to *am* as PM is to *pm*. And, conversely, if AM be always to *am* as AS is to *a*S; and PM be always to *pm* as AM is to *am*, it is manifest, that the points S, *p* and P are always in one right line, and that S*p* is to SP in the invariable ratio of S*a* to SA. From which it appears, that the common definition of similar figures, particularly that of APOLLONIUS for the conic sections, and of CAVALERIUS for any figures whatsoever, is included in this, or is easily deduced from it. When the similar figures are in the situation we have described, they are also similarly situated, and all their homologous lines are either placed upon one another or parallel.

123. If the right line SP, by revolving about the given point S, generate the similar figures ASP, *a*S*p*, their fluxions shall be to each other in the duplicate ratio of SP to S*p*, or of SA to S*a*; and, consequently, in an invariable ratio. Therefore, by the sixth general theorem, the similar figures ASP, *a*S*p* which are generated in the same time are in the same invariable ratio. The triangles ASP, *a*S*p* and the segments ANP, *anp* are in the same duplicate ratio of SA to S*a*, or of AP to *ap*. If we suppose the ray S*a* and the figure *a*S*d* to increase or decrease so as to remain always similar to a given figure ASD, the fluxion of the area *a*S*d* shall be to the fluxion of the inscribed triangle

aSd in the invariable ratio of that area to the inſcribed triangle; and the fluxion of the area *aSd* increaſes or decreaſes uniformly when the fluxion of S*a* is conſtant.

CHAP. IV.

Of the Fluxions of Solids, and of third Fluxions.

PROP. VI.

FIG. 26.

124. THe *fluxion of the ſolid that is generated by the revolution of the area* ADEF *about the axis* AD, *is meaſured by the cylinder that is generated by the rectangle* EG, *when the fluxion of the axis* AD *is meaſured by* DG.

The demonſtration is the ſame as that of the fourth propoſition; and it is eaſy to deduce corollaries from this propoſition ſimilar to thoſe which were inferred from the fourth.

PROP. VII.

125. *The fluxion of a ſolid that can be conceived to be generated by any plane ſurface moving parallel to itſelf, and perpendicular to a given axis, is meaſured by a priſm that has the generating ſurface for its baſe, and its altitude equal to the right line which meaſures the fluxion of the axis.*

By a priſm we underſtand, in this propoſition, any upright ſolid that can be generated by an invariable plane moving parallel to itſelf perpendicular to a given right line. When the generating figure is of an invariable magnitude, the propoſition is demonſtrated as in art. 80. The axis AD being ſuppoſed to flow uniformly,

ly, if the generating plane LM continually increaſe, it is evident that the ſolid muſt increaſe with an accelerated motion. Let this motion be ſuppoſed to be continued uniformly from the term when the plane LM cuts the axis in D, and to generate a ſolid equal to R in the ſame time that the axis by flowing uniformly acquires the augment DG. Then ſhall the ſolid R be equal to the priſm upon the baſe LM of the height DG: For, if the ſolid R be ſaid to be greater than that priſm, let AD be produced to P till *lm* (the ſection of the ſolid through P parallel to LM) be to LM as the ſolid R is to that priſm; and the ſolid R ſhall be equal to a priſm upon the baſe *lm* of the height DG. From this it follows, (by theor. 1.) that the motion with which the ſolid EBML flows, continued uniformly, would generate a ſolid equal to a priſm upon the baſe *lm* of the height DP, in the time that the axis acquires the augment DP by flowing uniformly. But the motion with which the ſolid flows when continually accelerated from the ſame term, generates in that time the fruſtum included betwixt the ſections LM and *lm*, which is leſs than the priſm upon the baſe *lm* of the height DP; and, if the motion with which the ſolid flows was continued uniformly from that term, it would generate in the ſame time a leſs ſolid than that fruſtum, (by ax. 1.) and therefore leſs than the priſm upon the baſe *lm* of the height DP. But theſe are contradictory; and therefore the ſolid R is not greater than a priſm upon the baſe LM of the height DG. In like manner it is ſhewn, that the ſolid R is not leſs than this priſm. And the demonſtration is extended to the other caſes of the propoſition in the manner that was deſcribed in art. 90. Therefore, the fluxion of the axis AD being repreſented by DG, the fluxion of the ſolid is accurately meaſured by a priſm upon the baſe LM of the height DG. And, converſely, when the fluxion of the ſolid is repreſented by that priſm, the fluxion of the axis is accurately meaſured by DG. Fig. 34.

126. Cor. I. Let the ſolid ALM be a pyramid that has its vertex in A, and let the ſection LM be triple of the ſquare of AD: Then ſhall the ſolid ALM be always equal to a cube deſcribed upon AD. Therefore the fluxion of the cube upon AD is accurately meaſured by a parallelopipedon upon a baſe that is triple Fig. 35.

of

of the ſquare of AD, of a height equal to DG that meaſures the fluxion of the ſide AD, or by a parallelopipedon upon the ſquare of AD of a height triple of DG. From which it follows, that, when four quantities are in continued proportion, and the firſt term is invariable, the fluxion of the ſecond term is to one third part of the fluxion of the fourth as the firſt term is to the third.

127. Cor. II. When the ſolid ALM is a pyramid as in the laſt corollary, and the motion with which AD flows is uniform, the fluxion of the pyramid is as the ſection LM, or as the ſquare of AD. In this caſe, the right lines AL, AM, *&c.* that form the ſolid angle at A, flow with uniform motions. The ſection LM, and the triangular ſides of the pyramid increaſe with motions that are accelerated uniformly, by prop. 2. cor. 4. But the ſolid ALM increaſes with a motion the acceleration of which increaſes uniformly; that is, the ſecond fluxion of the pyramid increaſes uniformly, and its third fluxion is invariable. In the ſame manner, when the ſide of a cube or the axis of a cone increaſes uniformly, the ſolid flows with a motion the acceleration of which increaſes uniformly.

Fig. 21. 128. In art. 93. we reſolved the trapezium DGHE (which is the increment of the triangle ADE that is generated while AD by increaſing uniformly acquires the augment DG) into two parts, *viz.* the parallelogram EG, which is generated in conſequence of the motion with which the triangle flows at the term when P comes to D, and the triangle EIH, which is generated in conſequence of the acceleration of that motion. In like manner, we are in the preſent caſe to reſolve the increment of the ſolid that is generated while the axis acquires the augment DG, into three parts. The firſt we conceive to be generated in conſequence of the motion with which the ſolid flows, at the term when its ſide or axis becomes equal to AD, ſuppoſed to be continued uniformly for that time. The ſecond part is conceived to be generated in conſequence of the acceleration of this motion ſuppoſed to be continued uniformly from the ſame term and for the ſame time. And the third part is what is generated in conſequence of the continual and uniform increaſe of this acceleration.

Fig. 35.

129. Theſe three parts may be diſtinguiſhed from each other in

in the following manner. Let the triangle MP*m*, by moving perpendicular to the right line AP, generate the pyramid APM*m*. While the point P describes DG, let the pyramid acquire the increment that is terminated by the planes ED*e*, HG*b* parallel to MP*m*; let EI and *ei* parallel to DG meet GH and *gb* in I and *i*, and *e*N parallel to EH meet H*b* in N. Then the frustum of the pyramid terminated by the planes ED*e*, HG*b* is resolved into three parts; the prism ED*e*IG*i*, the prism EIH*ei*N, and the pyramid *e*N*ib*. The first of these, the prism ED*e*IG*i*, measures the first fluxion of the pyramid ADE*e* when DG represents the fluxion of AD, by the seventh proposition. The second, *viz.* the prism EIH*ei*N, measures one half of the fluxion of the first, the altitude DG being supposed invariable. For the fluxion of the triangle ED*e*, or IG*i*, is measured by the parallelogram IN, (the fluxion of DE or GI being measured by IH,) by prop. 2. and the fluxion of the prism ED*e*IG*i* is therefore measured by a parallelopipedon NHE*e* upon the base IN of the altitude DG, which is double of the prism EIH*ei*N that is upon the same base IN and of the same altitude. Therefore the prism EIH*ei*N measures one half of the fluxion of the prism ED*e*IG*i*, or one half of the second fluxion of the pyramid ADE*e*. The last part of the frustum ED*e*HG*b*, *viz.* the pyramid *e*N*ib*, measures one sixth part of the fluxion of the parallelopipedon NHE*e*. For, completing the parallelogram N*i*R*b* and the parallelopipedon N*b*L*e*, the fluxion of the parallelogram IN is measured by the parallelogram NR, (by prop. 1. IH being invariable,) and the fluxion of the parallelopipedon NHE*e* is measured by the parallelopipedon N*b*L*e*, of which the pyramid *e*N*ib* is one sixth part. Therefore this pyramid measures one sixth part of the fluxion of the parallelopipedon NHE*e*, or of the third fluxion of the pyramid ADE*e*. FIG. 36.

130. Thus the solids ED*e*IG*i*, NHE*e*, N*b*L*e* respectively measure the first, second, and third fluxions of the pyramid ADE*e*, when AD flows uniformly and its fluxion is represented by DG. The three parts which constitute the frustum ED*e*HG*b* (or the increment of the pyramid that is generated while AD acquires the augment DG) are, the prism ED*e*IG*i*, the prism EHI*e*N*i*, and the pyramid *e*N*ib*; of which the first measures the first flu-

xion, the ſecond meaſures one half of the ſecond fluxion, the laſt meaſures one ſixth part of the third fluxion of the pyramid AD*Ee*; and, this part *eNib* being invariable, the pyramid has no fourth fluxion. The firſt of theſe is the ſolid that would have been generated by the motion with which the pyramid AD*Ee* flows, if it had been continued uniformly for the time in which P deſcribes DG. The ſecond part is what would have been produced more than the firſt, if the acceleration of the generating motion had been continued uniformly from the term when P comes to D, in the ſame manner as the triangle EIH (fig. 21.) is the ſpace that is generated in conſequence of the acceleration of the motion with which the triangle APM flows while P deſcribes DG, according to what was ſhewn in the 93d article. The third part *eNib* is what is generated in conſequence of the continual uniform increaſe of the acceleration of the generating motion, or of the uniform increaſe of the power by which we may conceive this acceleration to be produced. The firſt fluxion of the pyramid AD*Ee* is accurately expreſſed by the firſt of thoſe parts of its increment, and the other two are juſtly neglected in meaſuring it. The ſecond part ſerves for comparing the ſecond fluxion of the pyramid at the term when P comes to D with its ſecond fluxion at any other term, or with the ſecond fluxion of any other ſolid. The third part *eNib* ſerves for comparing the third fluxion of the pyramid with that of any other ſolid, or with the third fluxion of the ſame pyramid when the axis AD flows with a different motion.

131. The priſm ED*e*IG*i*, which meaſures the firſt fluxion of the pyramid AD*Ee*, is to the parallelopipedon NH*Ee*, which meaſures its ſecond fluxion, as one half of DE is to EK, or as one half of AD is to DG. The parallelopipedon NH*Ee* is to the parallelopipedon N*b*L*e*, which meaſures the third fluxion of the pyramid AD*Ee*, as NH is to N*b*, or as AD is to DG. And it is manifeſt, that theſe ratios, by diminiſhing DE, or by diminiſhing the increment of the pyramid ED*e*HG*b*, may become greater than any aſſignable ratio. Therefore the ratio of the quantity that meaſures the firſt fluxion of the pyramid AD*Ee* to that which meaſures its ſecond fluxion, and the ratio of the latter to that which meaſures its third fluxion, may become greater

greater than any aſſignable ratio, when the quantity that meaſures its firſt fluxion is continually diminiſhed. The pyramid AGH*b* is preciſely equal to the ſum of theſe four ſolids, the pyramid ADE*e*, the priſm ED*e*IG*i*, the priſm EHI*e*N*i*, and the pyramid *e*N*ib*. When DG is continually diminiſhed, the firſt of thoſe four ſolids approaches to the pyramid AGH*b*, ſo that their difference may become leſs than any aſſignable magnitude; but the ſum of the firſt two approaches much nearer to the pyramid AGH*b*: and the ratio of the difference betwixt this ſum and that pyramid, to the difference of the pyramids AGH*b* and ADE*e*, may become leſs than any aſſignable ratio. The ſum of the firſt three of thoſe ſolids approaches ſtill nearer to the value of the pyramid AGH*b*; and the ratio of the difference betwixt that ſum and this pyramid, to the difference of the ſum of the firſt two ſolids and the ſame pyramid, may become leſs than any aſſignable ratio. Thus we are led to approximations of various degrees of exactneſs by this method, as well as to accurate menſurations. For, in general, the value of a fluent at the beginning of any ſmall time being given, we approximate to its value at the end of that time, by adding to the former, firſt, the quantity that would have been produced in this time by the generating motion, if it had been continued uniformly from the beginning of the time, (which we ſuppoſe to meaſure the firſt fluxion of the fluent;) then, one half of the quantity that meaſures its ſecond fluxion, and one ſixth part of the quantity that meaſures its third fluxion. The approximation is the more accurate, the more we add of thoſe quantities together, in the order we have deſcribed them: but of this we ſhall treat more fully afterwards.

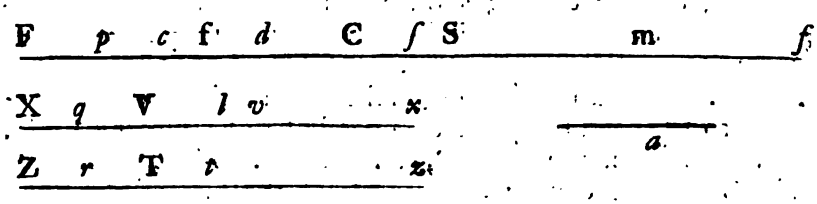
132. When AD increaſes uniformly, the priſm ED*e*IG*i* increaſes in the ſame proportion as the triangle ED*e*, (its altitude DG being invariable,) or in the duplicate ratio of AD; the parallelopipedon NHE*e* increaſes in the ſame proportion as E*e* or AD increaſes, becauſe the ſides HK and HI are invariable; the parallelopipedon N*b*L*e* is always of the ſame magnitude. But when AD is given, and DG, which meaſures the fluxion of AD, increaſes or decreaſes, the priſm ED*e*IG*i*, which meaſures the firſt fluxion of the pyramid ADE*e*, increaſes or decreaſes in

the same proportion; the parallelopipedon NHE*e*, which measures the second fluxion of that pyramid, increases or decreases in the same proportion as its base IK, or in the duplicate ratio of DG; and the parallelopipedon N*b*L*e*, which measures the third fluxion of the same pyramid, increases or decreases in the triplicate ratio of *ei*, or DG. Therefore, when the first fluxion of the pyramid at any given term is supposed to vary, the second fluxion varies in the duplicate ratio, and the third fluxion in the triplicate ratio of that in which the first fluxion varies.

FIG. 6. Pl. 2. 133. All we have said of the pyramid ADE*e* and its fluxions is easily transferred to the cone and its fluxions, by what was demonstrated in the introduction, (pag. 19. & 20.) The increment of the cone AH*b* that is generated while the axis AE is increased by EB, or the frustum described by the trapezium EBCH revolving about the axis EB, is equal to the cylinder YX*xy* (BX being a mean proportional betwixt BC and EH) and cone ES*s* (BS being the difference of BC and EH) taken together. The cylinder YX*xy* being resolved into two parts, the cylinder HZ*zb*, and the hollow cylindric solid that is generated by the rectangle HX revolving about the axis EB; the first of these measures the first fluxion of the cone AH*b*, when the axis flows uniformly and its first fluxion is represented by EB; and the latter part measures one half of the fluxion of the former, or one half of the second fluxion of the cone AH*b*. For, the fluxion of EH being represented by CZ, and the fluxion of its square by the rectangle contained by 2EH and CZ, (by prop. 2. cor. 6.) the difference of the squares of BX and BZ (or the rectangle contained by EH and CZ) measures one half of the fluxion of the square of EH; and the annular space described by ZX revolving about B measures one half of the fluxion of the circle described by BZ or EH. Therefore the hollow solid generated by the rectangle HX measures one half of the fluxion of the cylinder H*bz*Z, or one half of the second fluxion of the cone AH*b*. The cone ES*s* (which, with the cylinder YX*xy*, completes a solid equal to the frustum H*bc*C) measures one third part of the fluxion of the hollow solid described by the rectangle HX; because the fluxion of the excess of the square of BX above the square of BZ (or the fluxion

xion of the rectangle contained by EH and CZ) is measured by the square of BZ or of CS; and the fluxion of the solid generated by the rectangle HX is therefore measured by the cylinder that would be described by the rectangle ES revolving about EB; which cylinder is triple of the cone ES*f*. Thus it appears, that of the three solids which taken together are equal to the frustum H*bc*C, (the increment of the cone AH*b* that is produced while AE is increased by EB,) the cylinder H*bz*Z measures the first fluxion of the cone AH*b*, the hollow solid described by the rectangle HX measures one half of its second fluxion, and the cone ES*f* measures one sixth part of its third fluxion, the axis being supposed to flow uniformly and its fluxion being represented by DG.

134. Let the point *p* describe the right line F*f* in such a manner, that an upright parallelopipedon upon a given invariable base, as the square of any given line *a*, and of a height equal to F*p*, may be always equal to the pyramid APM*m*, (fig. 36.) and, the motion of *p* being then similar to that with which the pyramid flows, the acceleration of its motion shall increase uniformly, or the increments of its velocity generated in any equal times that succeed after each other shall always increase uniformly, when the motion of the point P is uniform. Let *d*S be described by *p* in any given time; let *d*C be the space that would be described in that time by it with its motion continued uniformly from *d*; and let *df* be the space that would have been described in the same time if the acceleration of its motion had been continued uniformly from that term. Then



shall upright parallelopipedons on the invariable base that is supposed equal to the square of *a*, and of the altitudes *d*C, C*f* and *f*S, be respectively equal to the prisms ED*e*IG*i*, EHK*N**i* and the pyramid *e*N*ib*; and the right lines *d*C, C*f* and *f*S shall measure

the

the first fluxion of F*d*, one half of its second fluxion, and one sixth part of its third fluxion, respectively.

135. Let the points *q* and *r* describe the right lines X*x* and Z*z* in such a manner that X*q* may be always equal to the space that would be described by *p*, and Z*r* may be equal to the space that would be described by *q*, if their motions were continued uniformly from the terms *p* and *q*, in the same time in which it is supposed that *p* would describe *d*C with its motion continued uniformly from *d*. Then the motion of *q* shall be uniformly accelerated, and the motion of *r* uniform, when the motion of P in the line AG (fig. 36.) is supposed uniform. Let *d*S, V*v* and T*t* be spaces described by the points *p*, *q* and *r* in the same time; and let V*l* be the space that would be described by *q* in that time by its motion continued uniformly from V. Then *d*C, V*l* and T*t* measure the first, second and third fluxions of F*d*, respectively. The upright parallelopipedons on the square of the invariable line *a*, of altitudes respectively equal to *d*C, V*l* and T*t* are equal to the solids ED*e*IG*i*, NHE*e* and N*h*L*e*; C*f* is one half of V*l*, and *f*S is one sixth part of T*t*; so that *d*S is equal to the sum of *d*C, one half of V*l* and one sixth part of T*t*, taken together.

136. Because *f*S is the difference of *d*S and *df*, and is equal to one sixth part of T*t*; it follows, that one sixth part of the third fluxion of the fluent F*p* is measured, at any term, by the difference of the increments that are produced in a given time, when the acceleration of the generating motion is continued uniformly, and when the increase of this acceleration is continued uniformly, from that term; and it is determined by a similar difference when the generating motion is continually retarded, or when its acceleration decreases.

Fig. 37. 137. The rest remaining as in the 129th article, let BD be equal to DG, and the plane OB*o* parallel to ED*e* meet AE, A*e*, EI, *ei* and *e*N in the points O, *o*, Y, *y* and *n*. Then, the frustum ED*e*OB*o* being equal to the excess of the sum of the prism ED*e*YB*y* and pyramid *eoyn* above the prism EYO*eyn*, it follows, that the difference between the frustums ED*e*HG*h* and ED*e*OB*o* is double of the prism EIH*ei*N, and therefore measures the second fluxion of the pyramid ADE*e*, when AD flows uniformly and

and its fluxion is repreſented by DG. It is alſo manifeſt, that the fruſtum HG*b*OB*o* exceeds the priſm IG*i*YB*y* by a ſolid that is equal to the ſum of the equal pyramids *ei*N*b* and *eony*; and therefore the difference between that fruſtum and priſm meaſures one third part of the third fluxion of the pyramid ADE*e*.

138. In the ſame manner, (fig. art. 134.) if f*d* and *d*S be deſcribed by *p* in equal times, and *dc* be equal to *d*C, then the difference between *d*S and *d*f is equal to 2C*ſ*, or to V*l*; and the difference of fS and *c*C is double of *ſ*S, or equal to one third part of T*t*. The former difference meaſures the ſecond fluxion of F*d*, and the latter meaſures one third part of its third fluxion, when *d*C meaſures its firſt fluxion. From which another theorem may be deduced for determining the third fluxions of quantities. If f*d*, *d*S and Sm be deſcribed by the point *p* in equal times that ſucceed after each other, then it is eaſy to ſee that T*t* is equal to the exceſs of the difference between Sm and *d*S above the difference between *d*S and f*d*. In other caſes, the ratio of T*t* to that exceſs, or of one third of T*t* to the difference between fS and *c*C, may not be a ratio of equality; but it approaches to that ratio, when thoſe increments are continually diminiſhed, as its limit.

139. It appears, from what was ſhewn in the introduction, that the third fluxion of any ſolid that can be generated by a conic ſection revolving on its axis is invariable, when the axis flows uniformly; the parabolic conoid excepted, which has its ſecond fluxion conſtant, and has no third fluxion. This account of third fluxions is ſimilar to that we gave of ſecond fluxions in art. 97. We have deduced thoſe illuſtrations of them from the common geometry, becauſe what relates to ſecond and third fluxions has been repreſented as very abſtruſe, and becauſe it is the application of this part of the method that has been found to be moſt liable to miſtakes.

CHAP.

CHAP. V.

Of the Fluxions of Quantities that are in a continued geometrical Progression, the first term of which is invariable.

140. Let the right lines E*e*, F*f* intersect each other at right angles in A; and, AS, AP and AL being in geometrical progression, let SA be taken upon AF, AP upon AE, and AL upon A*f*; and SPL shall be always a right angle. Let SA be invariable; and, while the point P with an uniform motion describes any equal lines pP, P*p*, let the point L describe lL and L*l*; produce lp and *lp* till they meet LP produced in *d* and D. Then, because the angle SP*d* is equal to Sp*d*, the angle PSp is equal to L*d*l. But, since SPL is a right angle, the angle SPp is equal to *d*Ll. Therefore the triangles PSp, L*d*l are similar. In like manner, the triangles PS*p*, LD*l* are similar. Therefore L*l* is to P*p* as DL is to SP, and Ll is to Pp as *d*L is to SP; so that L*l* is to Ll as DL is to *d*L. But the the angle PSD is equal to D*pe*, or *p*SA, and the angle PS*d* is equal to P*pd*, or pSA; and, consequently, the angle PSD exceeds PS*d* by DS*d*, equal to *p*Sp: so that DL being greater than *d*L, L*l* is greater than Ll. Therefore, when the motion of P is uniform, the spaces described by L in any equal succeeding times, perpetually increase, and its motion is accelerated.

FIG. 38.

141. Because the angle *p*SD is equal to *p*PD, or PSA, it is less than *p*S*l*, and *p*D is less than *pl*. Therefore D*l* is less than 2*pl*. But the angle pS*d* being equal to *d*P*p*, or PSA, it is greater than pSl; and, p*d* being therefore greater than pl, *d*l is greater than 2pl.

142. The velocity of L is to the velocity of P as 2AP is to SA. For, if the ratio of the velocity of L to the velocity of P be a greater ratio than that of 2AP to SA, let it be the same as that of 2A*p* (any quantity greater than 2AP) to SA, or of 2*pl* to S*p*. Then, because D*l* is less than 2*pl*, the velocity of L shall be to the velocity of P in a greater ratio than D*l* is to

S*p*,

Sp, or (the triangles S*p*P, D*l*L being ſimilar) L*l* is to P*p*. But, the motion of L being accelerated, while the motion of P is uniform, it follows from the firſt axiom, that L*l* is greater than the ſpace which would be deſcribed by L if its motion was continued uniformly while P deſcribes P*p*. Therefore the velocity of L is to the velocity of P in a leſs ratio than L*l* is to P*p*. And theſe being contradictory, it follows, that the velocity of L is not to the velocity of P in a ratio greater than that of 2AP to SA. Let the ratio of thoſe velocities be leſs than that of 2AP to SA, and be the ſame as that of 2Ap to SA, or of 2pl to Sp; which being leſs than that of *d*l to Sp, (becauſe *d*l is greater than 2pl,) the velocity of L ſhall be to the velocity of P in a leſs ratio than that of Ll to Pp. But it follows from the ſecond axiom, that Ll is to Pp in a leſs ratio than the velocity of L is to the velocity of P. And theſe being contradictory, it appears, that the velocity of L is to the velocity of P, or the fluxion of AL to the fluxion of AP, as 2AP is to SA.

143. Let the angles SPL, PLM, LMN, MNR, &c. be always right, and the angular points P, L, M, N, R, &c. deſcribe always the right lines AE, A*f*, A*e*, AF, AE, &c. Then ſhall SA, AP, AL, AM, AN, AR, &c. be always a ſeries of quantities in a continued geometrical progreſſion, the firſt term of which SA is invariable. While the point P deſcribes any equal lines pP, P*p* on the line AE, let the points L, M, N, R, &c. deſcribe the ſpaces lL and L*l*, mM and M*m*, nN and N*n*, rR and R*r*, &c. on the lines A*f*, A*e*, AF, AE, the angles Spl and S*pl*, plm and *plm*, lmn and *lmn* being always right. Let lp and *lp* produced meet LP produced in *d* and D; let ml and *ml* produced meet ML in *g* and G; let nm and *nm*, rn and *rn*, &c. produced meet NM, RN, &c. in *h* and H, *k* and K, &c. reſpectively. Then it is manifeſt, that the triangles PSp, L*d*l, M*g*m, N*h*n, &c. and the triangles PS*p*, LD*l*, MG*m*, NH*n*, RK*r*, &c. are each a ſeries of ſimilar triangles. It is alſo evident, that the angles *p*SA, PSD, LDG, MGH, NHK, &c. and the angles pSA, PS*d*, LD*g*, MG*h*, NH*k*, &c. are each a ſeries of equal angles; and that any angle of the former ſeries exceeds always the correſponding angle of the latter ſeries by the difference *p*Sp. Fig. 39.

LEMMA VI.

144. *When the first term of a geometrical progression is invariable, and the second term increases uniformly, each of the following terms increases with an accelerated motion.*

Because the triangles PSp, L*d*l, M*g*m, N*h*n, R*k*r, &c. are similar, the increments pP, lL, mM, nN, rR, &c. are in the same proportion as the right lines SP, *d*L, *g*M, *h*N, *k*R, &c. and, the triangles PS*p*, LD*l*, MG*m*, NH*n*, &c. being also similar, the increments P*p*, L*l*, M*m*, N*n*, R*r*, &c. are in the same proportion as the right lines SP, DL, GM, HN, KR, &c. Therefore, pP being equal to P*p*, lL is to L*l* as *d*L is to DL, mM is to M*m* as *g*M is to GM, nN is to N*n* as *h*N is to HN, rR is to R*r* as *k*R is to KR. But DL is greater than *d*L, GM is greater than *g*M, HN is greater than *h*N, KR is greater *k*R, and so on; because the angles PSD, LDG, MGH, NHK, &c. exceed the angles PS*d*, LD*g*, MG*h*, NH*k*, &c. respectively, (the excess being always equal to the angle *p*Sp.) Therefore L*l*, M*m*, N*n*, R*r*, &c. exceed Ll, Mm, Nn, Rr, &c. respectively; and, the motion of the point P being uniform, (so that the equal lines pP, P*p* may be described by it in equal times,) the motions of the points L, M, N, R, &c. are perpetually accelerated.

145. It is manifest, that the lines D*l*, G*m*, H*n*, K*r*, &c. are respectively less than 2*pl*, 3*lm*, 4*mn*, 5*nr*, &c. but that the lines *d*l, *g*m, *h*n, *k*r, &c. are respectively greater than 2pl, 3lm, 4mn, 5nr, &c. For, the angles *p*SD, *l*DG, *m*GH, *n*HK, &c. being always equal to each other and to the angle PSA, and the angles *p*S*l*, *lpm*, *mln*, *nmr*, &c. being also equal, the latter always exceed the former by the angle PS*p*. Therefore, as *pl* is greater than *p*D, and D*l* therefore less than 2*pl*; so G*l* is less than 2*lm*, and G*m* less than 3*lm*; H*m* is less than 3*mn*, and H*n* less than 4*mn*; K*n* is less than 4*nr*, and K*r* is less than 5*nr*. In the same manner, the angles pS*d*, l*dg*, m*gh*, n*hk*, &c. being always equal to PSA, and the angles pSl, lpm, mln, nmr, &c. being equal,

Fig. 32 N. 1.

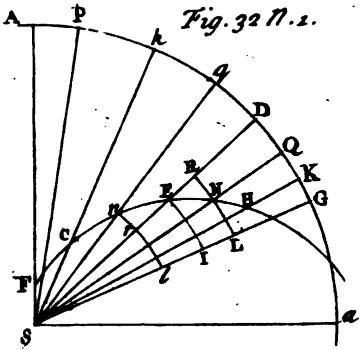


Fig. 32. N. 2.

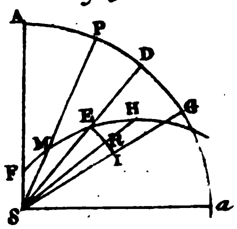


Fig. 34

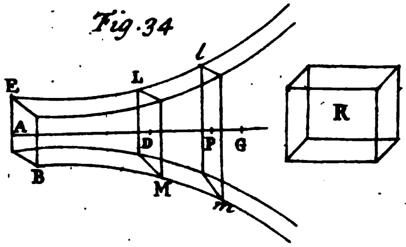


Fig. 35

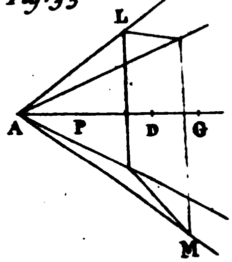


Fig. 37.

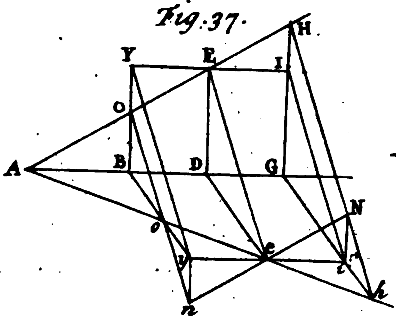
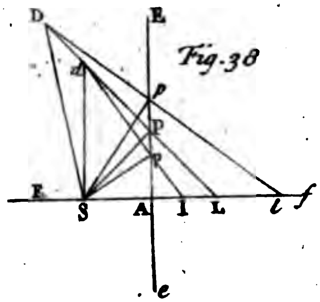


Fig. 38



qual, the former angles always exceed the latter by the angle PSp. Therefore, as p*d* is greater than pl, and consequently *d*l is greater than 2pl; so l*g* is greater than 2lm, and *g*m greater than 3lm; m*b* is greater than 3mn, and n*b* greater than 4nm; n*k* is greater than 4nr, and *k*r greater than 5nr; and so on.

P R O P. VIII.

146. *The fluxion of any term* AN *of a geometrical progression, the first term of which is invariable, is to the fluxion of the second term* AP *in a ratio compounded of the ratio of those terms and the ratio of the number of terms which preceed* AN *to unit.*

The fluxion of any term AN is to the fluxion of the second term AP, in a ratio compounded of the ratio of AN to AP, and of the ratio of the number of terms that preceed AN to unit; that is, in this example, the fluxion of AN is to the fluxion of AP as 4AN is to AP. First, let the motion of P be uniform, and the motion of N shall be accelerated, by lemma 6. Then, if the ratio of the velocity of N to the velocity of P be not that of 4AN to AP, or of 4AM to SA, let it first be the same as that of 4A*m* to SA, A*m* being any line greater than AM. The triangles SA*p*, A*mn* being similar, 4A*m* is to SA as 4*mn* is to S*p*; but 4*mn* is greater than H*n*, (by art. 145.) and 4*mn* is to S*p* in a greater ratio than H*n* is to S*p*, or N*n* to P*p*. Therefore the ratio of the velocity of N to the velocity of P is greater than the ratio of N*n* to P*p*. But the motion of N being accelerated while the motion of P is uniform, it follows from the first axiom, that a less space than N*n* would be described by the motion of N continued uniformly in the time P*p* is described by P. Therefore the velocity of N is to the velocity of P in a less ratio than N*n* is to P*p*. But these are contradictory, and therefore the ratio of the velocity of N to the velocity of P is not greater than that of 4AM to SA, or of 4AN to AP. Let the ratio of those velocities be that of 4Am to SA, Am being any line less than AM. Then, 4Am being to SA as 4mn is to

 Sp,

Sp, and 4mn being less than *bn*, (by art. 145.) it follows, that the velocity of N is to the velocity of P in a less ratio than *bn* is to Sp, or Nn is to Pp. But, by the second axiom, a greater space than Nn would be described by the motion of N continued uniformly, in the time pP is described by P; and therefore the velocity of N is to the velocity of P in a greater ratio than Nn is to Pp. But these being also contradictory, the velocity of N is to the velocity of P neither in a greater nor in a less ratio than that of 4AM to SA, or of 4AN to AP; and therefore the fluxion of AN is to the fluxion of AP precisely in the same ratio as 4AN is to AP. All the other cases of this proposition, when the motion of P is variable, are easily deduced from this case, by the eleventh general theorem. And it is obvious, that this demonstration is applicable in the same manner to any other terms of the progression.

PROP. IX.

147. *The fluxions of any two terms in a geometrical progression, the first term of which is invariable, are in a ratio compounded of the ratio of those terms to each other, and of the ratio of the numbers that shew how many terms preceed them in the progression.*

This follows from the last proposition. The fluxion of AN, for example, is to the fluxion of AM as 4AN is to 3AM. For the fluxion of AN is to the fluxion of AP as 4AN is to AP, by the last proposition; and the fluxion of AM is to the fluxion of AP as 3AM is to AP, by the same. Therefore the fluxion of AN is to the fluxion of AM as 4AN is to 3AM; and it is manifest, that the same demonstration may be applied when the fluxions of any other terms of the progression are compared together.

148. It follows from what has been demonstrated, that, if the motions of the points P, L, M, N, R, &c. were continued uniformly, lines respectively equal to SA, 2AP, 3AL, 4AM, 5AR, &c. would be described by them in the same time.

149. The

149. The ſecond, third, and higher fluxions of any term of the progreſſion, may be repreſented by certain multiples of the preceeding terms when the firſt fluxion of the ſecond term AP is conſtant (or the motion of P is uniform) and is repreſented by the invariable right line SA. For, the velocity of the point N, or the firſt fluxion of AN, being repreſented by 4AM; the fluxion of this velocity, or the ſecond fluxion of AN, may be repreſented by four times the fluxion of AM, (which is itſelf meaſured by 3AL,) or by 12AL; the third fluxion of AN is repreſented by 24AP, (becauſe the fluxion of AL is repreſented by 2AP,) its fourth fluxion by 24SA, and it has no fifth fluxion.

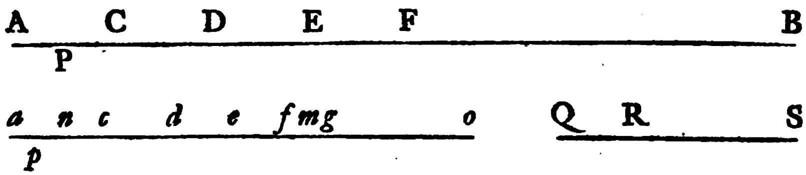
150. If A*u* be a mean proportional between AL and A*l*, and AP flow uniformly, then is 2L*u* equal to the line that would be generated by the motion with which AL flows continued uniformly for the time in which P deſcribes P*p*; and the difference betwixt *lu* and L*u* is what is generated in conſequence of the acceleration of this motion during that time. For A*u* is to AL in the ſubduplicate ratio of A*l* to AL, or in the ratio of A*p* to AP; and L*u* is to P*p* as AL is to AP, or as AP is to SA. Therefore 2L*u* is to P*p* as 2AP is to SA, or as the velocity of the point L is to the velocity of P. Let A*b* and A*k* be two mean proportionals between AM and A*m*; and, if the motion of M was continued uniformly while P deſcribes P*p*, a line would be deſcribed by it equal to 3M*b*; if the acceleration of that motion was continued uniformly from the ſame term and for the ſame time, a line would be deſcribed by it equal to 3*bk*; and what would be generated in conſequence of this uniform acceleration is equal to thrice the difference betwixt *bk* and M*b*. The difference betwixt M*m* and 3*bk*, or the exceſs of the difference betwixt *km* and *bk* above the difference betwixt *bk* and M*b*, is what is generated in conſequence of the increaſe of that acceleration during the ſame time. The proportions betwixt thoſe differences and the lines that meaſure the firſt, ſecond and third fluxions of AM, eaſily appear, and have been already ſhewn in the laſt chapter. Any term has fluxions of as many degrees as there are terms that preceed it in the progreſſion, when the fluxion of AP is invariable; and the increment of any term that is generated in a given time, may be reſolved in this manner into FIG. 40.

to as many parts as it has fluxions of different orders; and each part may be conceived to be generated in consequence of its respective fluxion. But this will appear more easily afterwards.

CHAP. VI.

Of Logarithms, and the Fluxions of logarithmic Quantities.

151. THe Logarithms were invented for facilitating computations in Arithmetic and Trigonometry, by the celebrated Lord NAPIER, Baron of Merchiston, and have been found since to be of great use in the higher Geometry, particularly in the method of Fluxions. Their nature and genesis is proposed by the inventor in a method similar to that which is applied in this doctrine for explaining the genesis of quantities of all sorts, and is described by him almost in the same terms. He begins his treatise on this subject, by defining that a line increases equally, when the point that describes it moves over equal spaces in equal times. Let A (says he *) be the term from which the line is to be described by the flux or motion of



the point P. Let it flow from A to C in the first moment, (or in any small part of the time,) from C to D in the second moment,

* *Sit punctus* A *à quo ducenda sit linea fluxu alterius puncti qui sit* P. *Fluat ergo primo momento* P *ab* A *in* C, *secundo momento à* C *in* D, &c. Mirif. Logar. Canon. descript. defin. 1. He afterwards lays down a postulate, similar to what we assumed in the first chapter, in these words: *Quum quolibet motu & tardior & velocior dari possit, sequetur necessariò cuique motui æquivelocem (quem nec tardiorem nec velociorem definimus) dari posse.*

ment, from D to E in the third; and ſo on for ever, deſcribing always the equal parts AC, CD, DE, EF, *&c.* in equal times. This line is then ſaid to increaſe *equally*.

152. By his ſecond definition *, a line decreaſes *proportionally*, when the point that moves over it deſcribes ſuch parts in equal times as are always in the ſame conſtant ratio to the lines from which they are ſubducted, or to the diſtances of that point at the beginning of thoſe times from a given term in the line. Let the ratio of QR to QS be any given ratio; let *ac* be to *ao*, *cd* to *co*, *de* to *do*, *ef* to *eo*, *fg* to *fo*, *&c.* always in the ſame invariable ratio of QR to QS. Suppoſe that the point *p* ſets out from *a*, deſcribing *ac*, *cd*, *de*, *ef*, *fg*, *&c.* in equal parts of the time; and let the ſpace deſcribed by *p* in any given time be always in the ſame ratio to the diſtance of *p* from *o* at the beginning of that time. Then the right line *po* is ſaid to decreaſe proportionally. It is manifeſt, that the lines *ao*, *co*, *do*, *co*, *fo*, *&c.* or the diſtances of the point *p* from *o* at equal ſucceeding intervals of time, are in a continued geometrical progreſſion.

153. Suppoſe now that the uniform motion of the point P in deſcribing the line AB is equal to the motion with which *p* ſets out from *a* in deſcribing the line *ao*; and the line AP that is deſcribed by P with this uniform motion, in the ſame time that *oa* by decreaſing proportionally becomes equal to *op*, is the *Logarithm* of *op* †. Thus AC, AD, AE, AF, *&c.* are the logarithms of *oc*, *od*, *oe*, *of*, *&c.* reſpectively; and *oa* is the quantity whoſe logarithm is ſuppoſed equal to nothing.

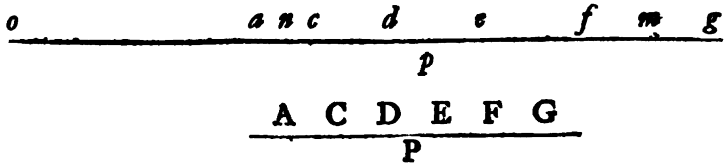
154. In like manner, ſuppoſe the line *oa* to increaſe proportionally; that is, let the point *p* move in the line *oa* produced beyond *a*, and in any equal times deſcribe ſpaces proportional to its diſtances from *o* at the beginning of each time; ſo that

* *Linea proportionaliter in breviorem decreſcere dicitur, quum punctus eam tranſcurrens æqualibus momentis ſegmenta abſcindit ejuſdem continuò rationis ad lineas à quibus abſcinduntur.* Ibid. defin. 2.

† *Logarithmus cujuſque ſinûs eſt numerus quàm proximè definiens lineam quæ æqualiter crevit, interea dum ſinûs totius linea proportionaliter in illum ſinum decrevit, exiſtente utroque motu ſynchrono, atque initio æquiveloce*, Ibid. That this doctrine may be more general, we abſtract from numbers, and repreſent the logarithms by lines, as well as the quantities of which they are the logarithms.

the

the increments *ac*, *cd*, *de*, *ef*, &c. may be deſcribed by it in equal times, when *ac* is to *ao*, *cd* to *co*, *de* to *do*, *ef* to *eo*, &c. always in the ſame invariable ratio. When *p* ſets out from *a*, let P ſet out from A with an equal velocity; then, the motion of P being continued uniformly while *op* increaſes proportionally, the line AP is always the logarithm of *op*; and the lines AC, AD, AE, AF, &c. are the logarithms of *oc*, *od*, *oe*, *of*, &c. re-



ſpectively. It is manifeſt, that the lines *oa*, *oc*, *od*, *oe*, *of*, &c. are in continued geometrical proportion. If *ac* and *fg* be deſcribed by *p* in equal intervals of time, and *an*, *fm* be deſcribed by it in any equal parts of thoſe times; then ſhall *ac* be to *fg*, and *an* to *fm*, as *oa* is to *of*, or as *oc* is to *og*, or as *on* is to *om*; that is, the ſpaces deſcribed by *p* in equal times are in the ſame proportion to each other as the diſtances of *p* from *o*, at the beginning, end, or any ſimilar term of thoſe times.

155. When a ratio is given, the point *p* deſcribes the difference of the terms of the ratio in the ſame time. When a ratio is duplicate of another ratio, the point *p* deſcribes the difference of the terms in a double time. When a ratio is triplicate of another, it deſcribes the difference of the terms in a triple time; and ſo on. Thus the ratio of *od* to *oa* is duplicate of the ratio of *oc* to *oa*; and, the lines *ac*, *cd* being deſcribed by *p* in equal times, the time in which *ad* is deſcribed by it is double of that in which it deſcribes *ac*. The ratio of *oe* to *oa* is triplicate of the ratio of *oc* to *oa*, and the time in which *ae* is deſcribed by *p* is triple of that in which it deſcribes *ac*. In the ſame manner it appears, that, when a ratio is compounded of two or more ratios, the point *p* deſcribes the difference of the terms of that ratio, in a time equal to the ſum of the times in which it deſcribes the differences of the terms of the ſimple ratios of which it is compounded.

156. What

156. What we have ſaid of the times of the motion of *p* when *op* increaſes proportionally, is to be applied to the ſpaces that are deſcribed by P in thoſe times with its uniform motion; and hence the chief properties of the logarithms are deduced. The difference of the logarithms of the terms of the ſame ratio is always the ſame. When a ratio is duplicate of another ratio, the difference of the logarithms of the terms is double; when a ratio is triplicate, that difference is triple; and ſo on. The difference of the logarithms of the terms of a compound ratio, is equal to the ſum that is produced by adding together the differences of the logarithms of the terms of the ſimple ratios from which it is compounded.

157. Thus logarithms are the meaſures of ratios. The exceſs of the logarithm of the antecedent above the logarithm of the conſequent meaſures the ratio of thoſe terms. The meaſure of the ratio of a greater quantity to a leſſer is poſitive, as this ratio compounded with any other ratio increaſes it. The ratio of equality compounded with any other ratio neither increaſes nor diminiſhes it; and its meaſure is nothing. The meaſure of the ratio of a leſſer quantity to a greater is negative, as this ratio compounded with any other ratio diminiſhes it. The ratio of any quantity A to unit compounded with the ratio of unit to A produces the ratio of A to A, or the ratio of equality; and the meaſures of thoſe two ratios deſtroy each other when added together; ſo that, when the one is conſidered as poſitive, the other is to be conſidered as negative. By ſuppoſing the logarithms of quantities greater than *oa* (which is ſuppoſed to repreſent unit) to be poſitive, and the logarithms of quantities leſs than it to be negative, the ſame rules ſerve for the operations by logarithms, whether the quantities be greater or leſs than *oa*. From what we have ſaid, it is eaſy to ſee how the logarithms ſerve for abridging computations. Becauſe unit is to any quantity A as B is to the product of A and B, and the logarithm of unit is nothing, the logarithm of any product is equal to the ſum of the logarithms of the factors, and the logarithm of any power of A is to the logarithm of A as the exponent of that power is to unit.

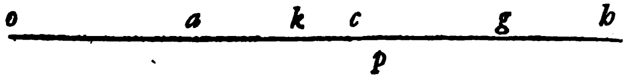
158. When *op* increaſes proportionally, the motion of *p* is

perpetually accelerated; for the spaces *ac*, *cd*, *de*, &c. that are described by it in any equal times that continually succeed after each other, perpetually increase in the same proportion as the lines *oa*, *oc*, *od*, &c. When the point *p* moves from *a* towards *o* as in the 152d article, and *op* decreases proportionally, the motion of *p* is perpetually retarded; for the spaces described by it in any equal times that continually succeed after each other, decrease in this case in the same proportion as *op* decreases.

PROP. X.

159. *The fluxion of a quantity that increases or decreases* proportionally, *varies always in the same ratio as the quantity itself.*

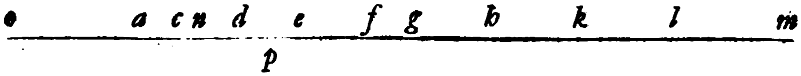
Let the line *op* increase or decrease proportionally, and the velocity of the point *p* (or the fluxion of *op*) shall always vary in the same ratio as its distance from *o*. Thus the velocity of *p* at the term *c* shall be to its velocity at any other term *g*, as *oc* is to *og*. For, if the ratio of the velocity of *p* at *g* to its velocity at *c* be greater than that of *og* to *oc*, let it be the same as that of any line *ob* greater than *og* to *oc*; and let *ok* be to *oc* as *og* is to *ob*. Then is *gb* to *kc* as *ob* is to *oc*, or as *og* is to *ok*; and the lines *kc*, *gb* are described by the point *p*



in the same time, by the supposition. Let *op* first increase proportionally: and, because the motion of *p* is then perpetually accelerated, (art. 158.) it follows from the first axiom, that a less line than *gb* would be described by it, if its motion was continued uniformly from *g*, in the same time that it describes *gb* while *op* increases proportionally; and it follows from the second axiom, that a greater line than *kc* would be described in an equal time by *p*, if its motion was continued uniformly from the term *c*. Therefore the velocity of *p* at *g* is to its velocity at *c* in a ratio that is less than the ratio of *gb* to *kc*, or of *ob* to *oc*. But it was supposed, that the velocity of *p* at *g* is to the velocity

locity of p at c as oh is to oc. And these being contradictory, it follows, that the velocity of p at g is not to its velocity at c in any ratio greater than that of og to oc. In the same manner it appears, that the velocity of p at c is not to its velocity at g in any ratio greater than that of oc to og; from which it follows, that the velocity of p at g is not to the velocity of p at c in any ratio that is less than the ratio of og to oc. Therefore the velocity of p at g is to the velocity of p at c precisely as og is to oc; and the fluxion of og is to the fluxion of oc in the same ratio. When op decreases proportionally, the demonstration is deduced in the same manner from the third and fourth axioms.

160. But it may be of use for illustrating the account of logarithms that is proposed by the inventor, to demonstrate the converse of this proposition from the axioms, and shew that when the velocity of p is always as the distance op, then this line increases or decreases in the manner that is supposed by him. Let the velocity of p therefore increase always in the same proportion as op increases; let af be to gm as oa is to og: and the spaces af, gm shall be described by the point p in equal times; that is, op shall increase proportionally. For, if the time in which af is described by p be supposed to be greater than the time in which gm is described by it, let on be greater than oa in the same ratio. Let oa, oc, od, oe, of be a series of lines in continued proportion, the number of proportionals between oa and of being increased till the term oc (which is next to oa) become less than on. Because oa is to of as og is to om, if og, oh,



ok, ol, om be also in continued proportion, and the number of terms be the same as in the other series, the lines ac, cd, de, ef, gh, hk, kl, lm shall be in the same proportion to each other respectively, as the lines oa, oc, od, oe, og, oh, ok, ol, or as the lines oc, od, oe, of, oh, ok, ol, om. Suppose first the line ac to be described by an uniform motion equal to that of p at the term a, cd to be described by an uniform motion equal to that of p at

 the

the term c; and dc, ef, gh, hk, kl, lm to be described by uniform motions equal to those of the point p at the respective terms d, e, g, h, k, l. Then, these spaces ac, cd, de, ef, gh, hk, kl, lm being in the same ratios to each other as the velocities of the uniform motions by which they are supposed to be described, it follows, that they are all described in this case in equal times; and, the times in which af and gm are thus described being equimultiples of the times in which ac and gh are described, they are therefore equal. Suppose now the same spaces ac, cd, de, ef, gh, hk, kl, lm to be described by uniform motions respectively equal to those of p at the terms c, d, e, f, h, k, l, m, and the times in which they are in this case described, are all equal to each other, for the same reason; and, consequently, the times in which af and gm are thus described are equal in this case also. The time in which af is described in the first of those two cases, is to the time in which it is described in the second, as the time in which the line ac is described in the first is to the time in which the same line is described in the second, (Elem. 15. 5.) or as the velocity of the uniform motion by which it is described in the second is to the velocity of the uniform motion by which it is described in the first case; that is, as oc is to oa. From which it follows, that the time in which af is described in the first case is to the time in which gm is described in the second (which is equal to the time in which af is described in the same case) in the same ratio of oc to oa. But, the motion of p being accelerated continually while op increases, the time in which ac is described by p is less than the time in which ac is described by an uniform motion equal to that of p at a according to the first supposition. In the same manner, the times in which cd, de and ef are described by p are less than the times in which those spaces are described by uniform motions respectively equal to the motions of p at the terms c, d, and e; and, consequently, af is described by p in a time less than that in which af is described in the first of the two cases that were supposed by us. In the same manner, the time in which gh is described by p is greater than that in which gh is described by an uniform motion equal to that of p at h; and the time in which gm is described by p is greater than the time in which gm is described in the second of

of those two cases. Therefore the time in which *af* is described by *p* is to the time in which *gm* is described by it, in a less ratio than the time in which *af* is described in the former case is to the time in which *gm* is described in the latter; that is, in a less ratio than *oc* is to *oa*; and, consequently, in a less ratio than *on* is to *oa*. But the times in which *af* and *gm* are described by *p* were supposed to be in the same ratio as *on* is to *oa*. And these being contradictory, it follows, that the time in which *af* is described by *p* is not greater than the time in which *gm* is described by it. In the same manner it appears, that the latter of those times cannot exceed the former. Therefore the lines *af* and *gm* are described by *p* in equal times when the velocity of *p* increases in the same proportion as *op* increases. A demonstration of the same kind is easily adapted to the case when *op* is supposed to decrease.

161. Cor. I. The fluxion of any quantity *op* is to the fluxion of its logarithm AP (fig. art. 151. & 154.) as *op* is to *oa*, which represents unit and has its logarithm equal to nothing. The motion of P, by which AP the logarithm of *op* is described, was supposed (in art. 153. & 154.) uniform, and equal to the velocity of *p* at *a*, *oa* being the quantity whose logarithm is nothing. Therefore, by this proposition, the velocity of *p* at any term of the time is to the constant velocity of P as *op* is to *oa*. When the logarithm AP is supposed to flow with a variable motion, *op* does not increase proportionally: but, from what we have shewn, it may be demonstrated by the eleventh general theorem, that the velocity of *p* is still to the velocity of P (*op* being always the line of which AP is the logarithm) as *op* is to *oa*.

162. Cor. II. When *op* increases proportionally, the increments that are generated in any equal times are accurately in the same ratio as the velocities of *p*, or the fluxions of *op*, at the beginning, end, or at any similar terms of those times. Thus *af* is to *gm* (fig. art. 160.) as *oa* is to *og*, or as *of* is to *om*, or as *oc* is to *ob*.

163. Cor. III. When *op* increases or decreases proportionally, the fluxions of this line of all the higher orders increase or decrease in the same proportion as the line itself increases or decreases;

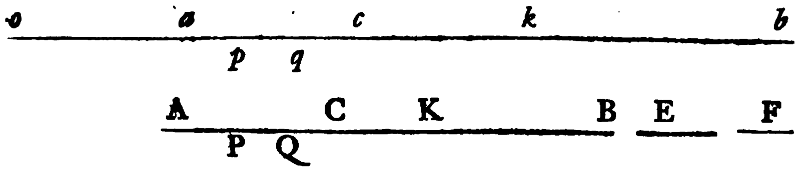
creases; so that one rule serves for comparing together those of any kind at different terms of the time. For, the velocity of *p* being always as *op*, it flows in the same manner as *op* flows, and its fluxion (or the second fluxion of *op*) varies in the same manner as the first fluxion of *op*, or as the fluent *op* itself varies. In the same manner, when the second fluxion of *op* is considered as a flowing quantity, its fluxion (or the third fluxion of *op*) is as the velocity of *p*, or as the line *op*. It is evident, that all the higher fluxions of *op* vary in the same manner; and that in this case we never arrive at any constant or invariable fluxion. In treating of those higher orders of fluxions, we never suppose any fluxion of *op* to be the velocity of a velocity; but we find, that the fluxion of any order flows in the same manner as *op* flows; and, considering that fluxion as a flowing quantity, its fluxion is as the velocity of *p* or as the line *op*. The first fluxion of any fluent is not the velocity of that fluent, but the velocity of the motion by which it is conceived to be generated; and, in like manner, the second fluxion of that fluent is not the velocity of the velocity of this fluent, but the velocity of the motion by which the quantity is generated that always represents its first fluxion: but of this we said enough above, in the first chapter, from art. 70. to the end, and in art. 97. 134. 135. 137. & 138.

PROP. XI.

165. *Let the logarithms of two quantities be always to each other in any invariable ratio, and the fluxions of those quantities shall be in a ratio that is compounded of the ratio of the quantities themselves and of the invariable ratio of their logarithms.*

Let the points Q and P set out from A together, and describe AB with uniform motions; and let the velocity of Q be to the velocity of P in the invariable ratio of E to F. Let the points *q* and *r* set out at the same time from *a* in the right line *ab* with velocities respectively equal to the velocities of Q and P. Then, if

if oq and op increase proportionally, AQ shall be the logarithm of oq, and AP the logarithm of op; and, AQ being to AP as the velocity of Q is to the velocity of P, or as E is to F, it follows, that the logarithm of oq is always to the logarithm of



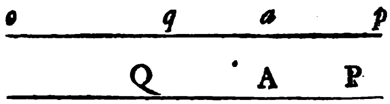
op in the invariable ratio of E to F. When p comes to c, let P, q and Q come to C, k and K respectively; and AK shall be to AC as E is to F. The velocity of q at k is to the velocity of q at a as ok is to oa, by prop. 10. The velocity of q at a is to the velocity of p at a as E is to F, by the supposition. The velocity of p at a is to the velocity of p at c as oa is to oc, by prop. 10. Therefore, by compounding those ratios together, the velocity of q at k is to the velocity of p at c as the rectangle contained by ok and E is to the rectangle contained by oc and F. But the fluxion of ok is to the fluxion of oc as the velocity of q at k is to the velocity of p at c; that is, in a ratio compounded of that of the lines ok and oc and of the invariable ratio of their logarithms AK and AC.

166. Cor. I. When oa, oc and ok are in continued geometrical progression, the logarithm of ok is always double of the logarithm of oc, by art. 156. and therefore the fluxion of ok is to the fluxion of oc as $2ok$ is to oc, or as $2oc$ is to oa, as has been already demonstrated in the 96th and 142d articles. When ok is any term of a continued geometrical progression of which oa and oc are the two first terms, and ok occupies always the same place in this progression; then, because the logarithm of ok is to the logarithm of oc as the number of terms that preceed ok in the progression is to unit, the fluxion of ok is to the fluxion of oc in a ratio compounded of the ratios of ok to oc and of the number of terms that preceed ok to unit, as was demonstrated in the eighth proposition. In general, when oc is any other term of the progression, the logarithm of ok is to the logarithm of oc

as

as the number of terms that preceed ok is to the number of terms that preceed oc; and the fluxion of ok is to the fluxion of oc in a ratio compounded of the ratio of those numbers and of the ratio of ok to oc, as we demonstrated in the ninth proposition in a different manner.

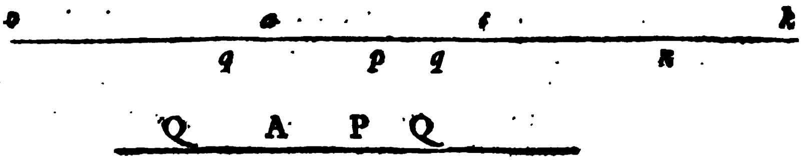
167. Cor. II. Let oq decrease proportionally while op increases; let the motions of the points Q and P be uniform, but in opposite directions from A, and be respectively equal to the motions of p and q at the term a: then the velocity of q shall be to the velocity of p in a ratio compounded of the ratio of oq to op and of the ratio of AQ to AP, as before: but the fluxion of oq must be considered in this case as negative, because oq decreases.



When op, oa and oq are in continued proportion, AQ the logarithm of oq is equal to AP the logarithm of op, the velocity of q is to the velocity of p as oq is to op; and any lines that measure the fluxions of the terms oq and op are in the same ratio as those terms themselves. In general, when oq, oa and op are terms of any geometrical progression, and the number of terms from oa to oq (including one of those terms themselves) is to the number of terms from oa to op (including one of these) as m is to n; then AQ is to AP as m is to n; and the velocity of q is to the velocity of p in a ratio compounded of the ratios of oq to op and of m to n. Any quantities that measure the fluxions of oq and op are in the same ratio; but the fluxion of oq is considered as negative when the fluxion of op is positive. Thus it appears how the fluxions of the terms of a geometrical progression are determined when any intermediate term of the progression is invariable.

168. Cor. III. In those cases, the velocity of Q is to the velocity of P, or the logarithm of oq to the logarithm of op, in a ratio which can be expressed by that of rational numbers. But the proposition is general, and the ratio of the fluxions of oq and op is assigned by it when the ratio of AQ to AP is invariable, though this ratio cannot be expressed by that of number to number. According to the notation which is now in use in Algebra, when AQ is to AP as m is to n, and oa being unit, oc is expressed

fed by A, and it is fuppofed (as in art. 165.) that oq becomes equal to ok when op becomes equal to oc, ok is expreffed by $A^{\frac{m}{n}}$ when the points p and q are on the fame fide of the point a, and by $A^{-\frac{m}{n}}$ when they are on oppofite fides of a. Whether the exponent $\frac{m}{n}$ be rational or irrational, the line that meafures the fluxion of ok is to that which meafures the fluxion of oc in a ratio compounded of the ratio of m to n and of the ratio of ok to oc. This notation by irrational exponents appears firft in Sir ISAAC NEWTON's letter to Mr. OLDENBURGH of October 24. 1676. If the ratio of AQ and AP, the logarithms of oq and op, be variable, let it be the fame as that of ox to oa, (or unit;)



and, op being expreffed by A, ox by x, oq may be expreffed by A^x, (the logarithms of powers and their exponents being always in the fame ratio,) and is called an *exponential* quantity of the firft degree by Mr. LEIBNITZ, who makes ufe of this notation in his anfwer to that letter. The rectangle contained by AQ and oa is equal to the rectangle contained by AP and ox; and hence rules for determining the fluxions of fuch exponential quantities may be deduced by the third and tenth propofitions: but we refer thefe to the fecond book.

169. Suppofe the logarithm AP to increafe uniformly; let pm be the fpace that would be defcribed by the motion of p continued uniformly, in the time P defcribes PG; and let op, pm, mn, nr, rs, &c. be a feries of terms in continued proportion. Then, if the fluxion of the logarithm AP be reprefented by PG, the firft, fecond, third, and all the higher fluxions of op fhall be reprefented refpectively by pm, mn, nr, and the fubfe- FIG. 41.

Y quent

quent terms of that geometrical progression. For, since *pm* is to *op* in the invariable ratio of PG to *oa*, it follows from the fifth general theorem, that the fluxion of *pm* is measured by a line which is to *pm* (that measures the fluxion of *op*) as PG is to *oa*, or as *pm* is to *op*: but *mn* is such a line, because *mn* is to *pm* as *pm* is to *op*; and therefore *mn* measures the fluxion of *pm*, or the second fluxion of *op*. In the same manner it appears, that *nr* measures the fluxion of *mn*, or the third fluxion of *op*; and that each higher fluxion is measured by a corresponding term of the progression. It is evident, that when *op* is determined, and *pm* (which measures the first fluxion of *op*) varies, then *mn* (which measures the second fluxion of *op*) varies in the duplicate ratio of *pm*; *nr* varies in the triplicate ratio of *pm*; and each term of the progression varies in the same proportion as the power of *pm* that has its exponent equal to the number of terms which preceed that term in the series.

170. We have shewn above, how to distinguish the increment of the fluent into such parts as may be conceived to be generated in consequence of its respective fluxions, and that bear a constant relation to their measures, in those cases when the fluent has no higher than first, second and third fluxions. In order to illustrate what relates to the higher orders of fluxions, (which is represented as very abstruse,) we shall now chuse an instance where fluxions of any order take place, and we never come to an invariable fluxion. Let *oq*, *oa* and *op* be in continued proportion; and, *op* being greater than *oa*, (which we suppose invariable,) let the motion of *q* be uniform: then
Fig. 42. the velocity of *p* shall increase in the duplicate ratio of *op*, because it is to the velocity of *q* as *op* is to *oq*, (art. 167.) or in the duplicate ratio of *op* to *oa*. While *q* describes *bl*, let *p* describe *ck*; and, *ob* being to *oa* as *oa* is to *oc*, and *ol* to *oa* as *oa* is to *ok*, it follows, that *ok* is to *oc* as *ob* is to *ol*, and *ck* to *oc* as *bl* is to *ol*. Let *cd* be to *dk*, *de* to *ek*, *ef* to *fk*, *fg* to *gk*, as *bl* is to *ol*; and the lines *cd*, *de*, *ef*, *fg*, *gx* shall be in a geometrical progression that may be continued at pleasure. The sum of the terms of this progression approaches continually to the increment *ck* as the series is produced, so that their difference may become less than any assignable line. The line *ob* being sup-

ſuppoſed to flow uniformly, let its fluxion be repreſented by *bl*; and the firſt, ſecond, third, fourth, and all the higher fluxions of *oc*, ſhall be repreſented reſpectively by *cd*, 2*de*, 6*ef*, 24*fg*, and the ſubſequent terms of the progreſſion having coefficients prefixed to them that ariſe by the continual multiplication of the numbers 1, 2, 3, 4, 5, 6, &c. The line *ck* which is the increment of the fluent *oc* conſiſts of the parts *cd*, *de*, *ef*, *fg*, *gx*, &c. and we approach continually to its value by adding to *cd*, which repreſents the firſt fluxion of *oc*, one half of the line that repreſents its ſecond fluxion, one ſixth part of the line that repreſents its third fluxion, one twenty fourth part of the line that repreſents its fourth fluxion, and ſo on. The firſt part *cd* is what would have been generated, in the time *p* and *q* deſcribe *ck* and *bl*, if the motion of *p* had been continued uniformly from the term when it comes to *c*; the ſum of the two firſt parts *cd* and *de* (*viz. ce.*) is what would have been generated in the ſame time, if the acceleration of the motion of *p* had been continued uniformly from the ſame term, as in the 74th and 97th articles; the ſum of the firſt three parts *cd*, *de* and *ef* (*viz. cf*) is what would have been generated in the ſame time if the increaſe of that acceleration had been continued uniformly, as in the 130th and 134th articles, or the line that repreſents the ſecond fluxion of *op* had increaſed uniformly; and ſo on. To demonſtrate theſe things, ſuppoſe *oa*, *oc*, *om*, *on*, *or*, &c. to be in continued proportion; then *cd* ſhall be to *om* as *bl* is to *oa*, *de* to *on* in the duplicate ratio of *bl* to *oa*, *ef* to *or* in the triplicate ratio of *bl* to *oa*, and any term of the ſeries *cd*, *de*, *ef*, *fg*, *gx*, &c. is to the correſponding term of the ſeries *om*, *on*, *or*, *os*, &c. in a conſtant ratio, becauſe *bl* and *oa* are invariable. Therefore, by the fifth general theorem, the fluxion of *cd* is to the fluxion of *om* as *cd* is to *om*; and, in general, the fluxions of the correſponding terms in thoſe two progreſſions are in the ſame ratio as the terms themſelves. But, by art. 166. the fluxion of *om* is to the fluxion of *oc* as 2*om* is to *oc*, the fluxion of *on* is to the fluxion of *oc* as 3*on* is to *oc*, the fluxion of *or* is to the fluxion of *oc* as 4*or* is to *oc*; and ſo on. Therefore the fluxion of *cd* is to the fluxion of *oc* (which is repreſented by *cd*) as 2*cd* is to *oc*; and, 2*de* being to *cd* in the ſame

ratio of $2cd$ to oc, it follows, that the fluxion of cd is represented by $2de$. The fluxion of de is to the fluxion of oc as $3de$ is to oc, and therefore is represented by $3ef$; and thus it appears, that the fluxions of oc, cd, de, ef, fg, &c. are represented respectively by cd, $2de$, $3ef$, $4fg$, $5gx$, &c. The second fluxion of oc is the same as the first fluxion of cd, and is therefore represented by $2de$. The third fluxion of oc is the same as the first fluxion of $2de$, and therefore is represented by $6ef$. The fourth fluxion of oc is the first fluxion of $6ef$, and is represented by $24fg$; and so on. While oq decreases, op increases; and, the fluxion of oq being considered as negative, all the fluxions of op are positive. When oq increases, op decreases; and, when p comes to c, all the fluxions of oc are represented by the same quantities as before; but they are alternately negative and positive when the fluxion of oq is invariable.

PROP. XII.

171. *Let op be greater than oa, ad to ap as oa is to op; and let oa, ad, de, ef, fg,* &c. *be in continued*
FIG. 43. *proportion: Then, by adding together ad, $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$,* &c. *we approximate continually to the value of* AP *the logarithm of op.*

Let op, oa and oq be in continued proportion, and the motion of q from a towards o be uniform, as in the last article. Then, because aq is to ap as oa is to op, ad is equal to aq, and the velocity of d is equal to the constant velocity of q. The velocity of P is to the velocity of p as oa is to op, (by art. 161.) and the velocity of p is to the velocity of q (or d) as op is to oq, (by art. 167.) Therefore the velocity of P is to the velocity of d as oa is to oq, or as op is to oa; so that the motion of P increases in the same ratio as op increases. The fluxion of AP is to the fluxion of ad in the same ratio of op to oa; and the terms oa, ad, de, ef, fg, &c. being in a continued geometrical progression, the first term of which oa is invariable, it follows from the eighth proposition, that the fluxion of ad is to the fluxions

xions of $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, &c. respectively, as *oa* is to *ad*, *de*, *ef*, &c. Therefore the fluxion of AP is to the fluxions of *ad*, $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, &c. respectively, as *op* is to the terms *oa*, *ad*, *de*, *ef*, &c. But, by adding together the terms *oa*, *ad*, *de*, *ef*, &c. we approximate continually to *op*; for *ad* is to *dp*, *de* to *ep*, *ef* to *fp*, &c. as *oa* is to *ap*; and the difference between *op* and the sum of the terms *oa*, *ad*, *de*, *ef*, &c. may become less than any assignable magnitude, by continuing the progression. Therefore, by adding together the fluxions of *ad*, $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, &c. we approximate continually to the value of the fluxion of AP; and, consequently, by summing up the lines *ad*, $\frac{1}{2}de$, $\frac{1}{3}ef$, $\frac{1}{4}fg$, &c. we approach continually to the value of AP the logarithm of *op*.

PROP. XIII.

172. *The same things being supposed as in the last proposition, we approximate continually to the logarithm of od by summing up the differences betwixt ad and $\frac{1}{2}de$, $\frac{1}{3}ef$ and $\frac{1}{4}fg$, $\frac{1}{5}gh$ and $\frac{1}{6}hk$,* &c.

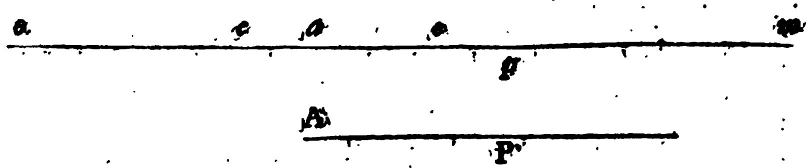
Let *od*, *oa* and *oz* be in continued proportion, and *oq* shall be to *oz* as *od* is to *op*; so that *qz* shall be to *dp* as *oq* is to *od*. Let *qz* be divided in the same proportion in the points *m*, *n*, *r*, *s*, &c. as *dp* is divided in the points *e*, *f*, *g*, *h*, &c. and the sum of the terms *oq*, *qm*, *mn*, *nr*, *rs*, &c. shall approximate continually to *oz*. But as the difference of *oa* and *ad* (or *aq*) is equal to *oq*, so the difference of *de* and *ef* is equal to *qn*: for *de* is to *ef* as *oa* is to *aq*, (or *ad*,) by the supposition; and, consequently, *df* is to the difference betwixt *de* and *ef* as *od* is to *oq*, or as *dp* is to *qz*, and therefore as *df* is to *qn*. In the same manner, the difference of *fg* and *gh* is equal to *ns*; and, in general, the difference of any two terms which immediately succeed each other in the progression *de*, *ef*, *fg*, *gh*, &c. is equal to the sum of the two corresponding terms of the progression *qm*, *mn*, *nr*, *rs*, &c. Therefore, by summing up the differences betwixt *oa* and *ad*, *de* and *ef*, *fg* and *gh*, &c. we approximate continually to the value of *oz*. But the fluxion of AD, the logarithm of *od*, is to the fluxion

xion of od (or of ad) as oa is to od, (by art. 161.) or as oz is to oa; and the fluxion of ad (by prop. 8.) is to the fluxions of the terms ad, $\frac{1}{3}de$, $\frac{1}{5}ef$, $\frac{1}{7}fg$, $\frac{1}{9}gh$, $\frac{1}{11}hk$, &c. respectively, as oa is to the terms oa, ad, de, ef, fg, gh, &c. Therefore we continually approximate to the value of the fluxion of AD by summing up the fluxions of the differences betwixt ad and $\frac{1}{2}de$, $\frac{1}{3}ef$ and $\frac{1}{4}fg$, $\frac{1}{5}gh$ and $\frac{1}{6}hk$, &c. and, consequently, we approximate to the value of AD, the logarithm of od, by summing up these differences themselves.

173. Cor. Let oa be to od as op is to ox; and the logarithm of ox shall be equal to the sum of the logarithms of op and od; to which therefore we approximate by summing up $2ad$, $\frac{2}{3}ef$, $\frac{2}{7}gh$, &c. It is manifest, that ox is to oa as od is to oq; and that the logarithm of ox is that which measures the ratio of od to oq. But od and oq have half their sum equal to oa and half their difference equal to ad, which are the two first terms of the geometrical progression oa, ad, de, ef, fg, gh, &c. and it is obvious, that what has been shewn of these logarithms, coincides with what the excellent Dr. Halley demonstrates in the Philosophical Transactions, num. 216. This would lead us to the method of computing logarithms, but we refer that to the next book.

174. Hitherto we have supposed, with Lord Napier in his first scheme of logarithms, that, while op increases or decreases proportionally, the uniform motion of the point P, by which the logarithm of op (or the measure of the ratio of op to oa) is generated, is equal to the velocity of p at a, that is, at the term of time when the logarithms begin to be generated. But the uniform motion of P may be supposed equal to the motion of p at any other term, as when it comes to e; in which case the constant velocity of P is to the velocity with which p sets out from a when the logarithms begin to be generated, as oe is to oa; and thus we may have as many systems of logarithms as we please. The properties mentioned in the 156th and 157th articles, by which they become useful for facilitating computations, are common to all the systems. The line oe is what the learned Mr. Cotes calls the *Modulus* of the system. The measures of a given ratio in the different systems are in the same propor-

proportion as the lines oe, e being always the term where the velocity of p becomes equal to the constant velocity of P. The logarithm of any quantity in NAPIER's first system becomes equal to the logarithm of the same quantity in any other system, whose modulus is oe, by multiplying it by the number which expresses the ratio of oe to oa; and the modulus of any system is to the modulus of any other system, as the logarithm of any given quantity in the first is to its logarithm in the second. Thus, in NAPIER's first scheme, in the same time that op from being equal to oa becomes equal to ten times oa, the point P describes a line the ratio of which to oa is that of 2.3025809 &c. to unit. But it was found convenient, that the logarithm of ten should be supposed equal to unit; and the motion of P was supposed to be so far diminished, that the space described by it in that time might be equal only to oa; that is, its velocity in this case was supposed less than its velocity in the former in the same ratio as 1 is less than 2.3025809 &c. If oe be taken less than oa in the same ratio, the velocity of P shall be equal in this case to the velocity of p at e; and oe shall be the modulus of this system, which therefore is expressed by 0.4342944 &c. oa being unit.



175. When a ratio is given, its measure is always as the modulus of the system. It is therefore the same invariable ratio that is always measured by the modulus of the system, which is by Mr. COTES called the *ratio modularis*. This ratio is that of om to oe, if op by increasing proportionally from being equal to oe become equal to om, in the same time that P by its uniform motion describes a line equal to the modulus oe.

176. While the base OP increases uniformly, let the ordinate Pp increase or decrease proportionally, (that is, let the velocity of p in the direction Op be always as the ordinate Pp;) and FIG. 44.

and the point p ſhall deſcribe the *logarithmic* curve. The baſe OP is always the logarithm of the ordinate Pp; and, if the uniform motion of P be equal to the motion of p at o in the direction Oo, then is the ordinate Oo the modulus of this ſyſtem of logarithms; and the fluxion of any ordinate is to the fluxion of the baſe as that ordinate is to Oo. The ordinates at equal diſtances from each other are always in geometrical progreſſion; from which it follows, that, when the baſe and curve are produced, they approach to each other continually on one ſide, ſo that their diſtance may become leſs than any aſſignable line; but, becauſe they can never meet, the baſe is therefore an aſymptote of the curve. When any two points of the curve, as a and d, are given, you may determine as many more points as you pleaſe, by raiſing ordinates upon the baſe at equal diſtances that are in any geometrical progreſſion of which the ordinates Aa and Dd are any two terms. When the curve is ſuppoſed to be deſcribed, the exponential quantities are eaſily determined by it. Let the ordinate Aa be expreſſed by A; let OP be to OA as any quantity expreſſed by x is to unit: and the ordinate Pp may be expreſſed by A^x, an exponential quantity of the firſt degree when x is variable.

FIG. 45. 177. Let lmn be an hyperbola, o its center, or and os its aſymptotes; let al, pm and gn parallel to the aſymptote or meet the curve in l, m and n. Let AL be equal to al, the angle LAP equal to lap, and AP be always the logarithm of op according to the 153d and 154th articles; then the hyperbolic area $almp$ ſhall be always equal to the parallelogram LP. For, if pg be to PG as the velocity of p is to the velocity of P, that is (by art. 161.) as op is to oa, or as al is to pm, (by the property of the hyperbola;) then ſhall pg be to PG as PM is to pm, the parallelogram mg ſhall be always equal to the parallelogram MG, and (by prop. 4.) the fluxion of the hyperbolic area $almp$ equal to the fluxion of the parallelogram LP. Therefore, by the fourth general theorem, the hyperbolic area $almp$ is equal to the parallelogram LP; and the area $almp$, or the ſector oml, (which is equal to it becauſe the triangles opm, oal are equal) meaſures the ratio of op to oa. Theſe areas form a ſyſtem of logarithms the modulus of which is the parallelogram ol, and ſerve

ſerve for meaſuring ratios, as the arches or ſectors of circles ſerve for meaſuring angles. The former are divided into equal parts, by reſolving a ratio into the equal ratios from which it is compounded; as, by dividing the latter into equal Parts, an angle is reſolved into the equal angles of which it conſiſts.

178. The area of the parabola admits of a perfect quadrature, and elliptic areas are reduced to ſuch as are circular; therefore the areas of all the conic ſections may be reduced to rectilineal figures, the meaſures of angles, or the meaſures of ratios: and to one or more of theſe all fluents ought to be reduced in this doctrine as much as poſſible. As a conic ſection paſſes from one ſpecies to another, by varying the inclination of the plane that cuts the cone; or by varying the circumſtances of the deſcription, when the curve is traced upon a plane by motion; or by changing a ſign, coefficient or exponent of a term, when the nature of the curve is expreſſed by an equation: ſo the expreſſions of the meaſures of angles and of ratios are by ſimilar variations transformed into each other, and in ſome caſes into ſuch as repreſent rectilineal figures; and, by a change in the ſign, coefficient or exponent of a term in the expreſſion of a fluxion, the nature of the fluent is ſo far altered that it becomes reducible to a conic ſection of a different kind. When *op* increaſes or decreaſes proportionally, the acceleration or retardation of the motion of *p* is as its diſtance from the given point *o*; but there are alſo other caſes in which its acceleration or retardation obſerves the ſame law: and, by taking theſe together, we may comprehend the geneſis of the lines that meaſure ratios and angles in one view. But we ſhall have occaſion afterwards to conſider further the analogy there is betwixt thoſe meaſures, and to treat of the logarithms that are called *imaginary*.

179. In NAPIER's firſt ſyſtem, the conſtant velocity of P, FIG. 46.
by which the logarithm AP is generated, is equal to the velocity with which *p* ſets out from *a* when their geneſis is ſuppoſed to begin, (art. 153. & 154.) Therefore, if we ſuppoſe AP and *ap* to be diminiſhed continually, (AP being always the logarithm of *op*,) their ratio ſhall approach continually to a ratio of equality as its limit, (by art. 67.) and, *oa* being ſuppoſed to repreſent unit, if *ap* be very ſmall compared with *oa*, it may be

supposed in approximations to be the logarithm of *op* in this system, which is called its *hyperbolic logarithm*; and any very small fraction may be supposed to be the logarithm of the sum of unit and that fraction added together. From this it follows, that, if a series of mean proportionals be interposed betwixt *oa* and any given line *ob*, and the number of all the terms without including *oa* be *x*, and *on* being the second term of the series *an* be to *ab* as unit is to the number *q*; then, by increasing continually the number of mean proportionals betwixt *oa* and *ob*, the ratio of *x* to *q* shall approach continually to the ratio of AB (the logarithm of *ob*) to *ab*, as its limit. For the number *x* is to unit as AB is to AN, (the logarithm of *on*;) and unit is to *q* as *an* (which approaches continually to AN) is to *ab*. Therefore the ratio of *x* to *q* approaches continually to that of AB to *ab*. For example, if *ob* be double of *oa*, the ratio of *x* to *q* * approaches continually to the ratio of the logarithm of 2 to unit, which is nearly that of 7 to 10.

CHAP. VII.

Of the Tangents of curve Lines.

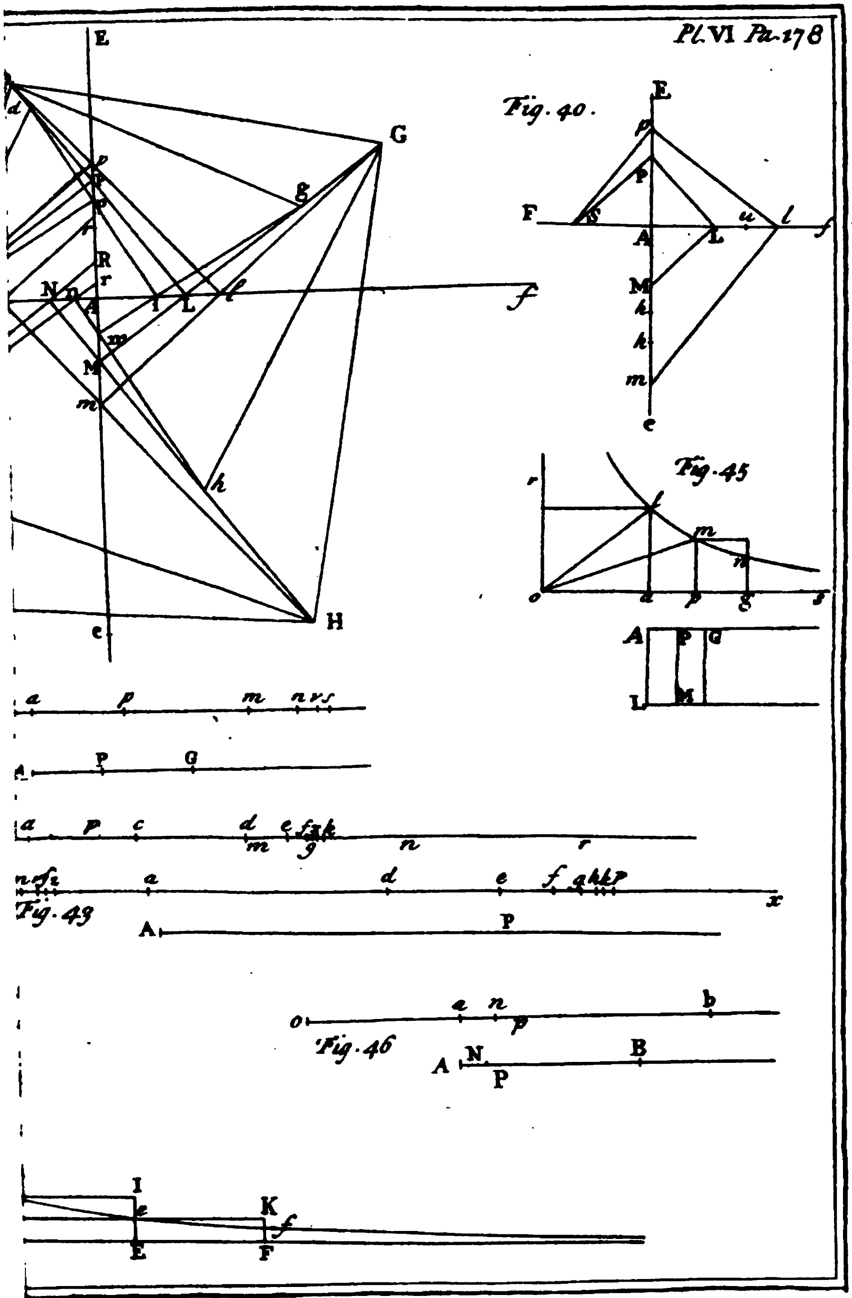
180. AN arch of a curve has its concavity turned one way, when the right lines that join any two of its points are all on the same side of the arch; or, in general, (that we may include, with ARCHIMEDES †, such lines as have rectilineal parts,) a line has its concavity turned one way, when the right lines that join any two of its points are either all upon one side of it, or while some fall upon the line itself, none fall on the opposite side.

* Thus, if the fraction $1 + \frac{1}{q}$ (represented here by *on*) raised to the power x be supposed equal to 2, the ratio of x to q shall be nearly that of the logarithm of 2 to unit, when q is a large number. See the *Doctrine of Chances, probl.* 5. by the excellent Analyst Mr. de Moivre.

† De sphæra & cylindro, defin. 2.

181. As

Pl. VI Pa. 178



181. As a right line is the tangent of a circle, when it touches the circle so closely that no right line can be drawn through the point of contact betwixt it and the arch, or within the angle of contact that is formed by them; so, in general, when a right line ET touches any arch of a curve, as EH in E, in such a manner that no right line can be drawn through E betwixt the right line ET and the arch EH, or within the angle of contact HET that is formed by them, then is ET the *Tangent* of the curve at E. It is manifest, that the right lines ET and EH are on different sides of the arch EH; and that when the arch has its concavity turned one way, the tangent at any point of it is on the convex side. FIG. 47.

182. The right line TE being continued to *t*, if E*t* is the tangent of the arch EC the continuation of HE, then the arch HEC has a continued curvature at E. When the arches EH and EC are on different sides of the tangent TE*t*, the point E is called a point of *contrary flexion*. But if any right line ER, different from E*t* the continuation of ET, touch the arch EC, then the point E is a *double point* of the curve, and is the intersection of two arches which have different tangents at that point, or are on opposite sides of the same tangent. FIG. 48. FIG. 49.

183. When two lines that have their concavity turned the same way have the same terms, and the one includes the other, or has its concavity towards it, the perimeter of that which includes is greater than the perimeter of that which is included. This is the second axiom of the treatise of ARCHIMEDES concerning the sphere and cylinder.

LEMMA VII.

184. *The base being supposed to flow uniformly, the ordinate increases with a motion that is continually accelerated, and decreases with a motion that is continually retarded, when the arch is convex towards the base. But when the arch is concave towards the base, the ordinate increases with a retarded motion, and decreases with an accelerated motion.*

FIG. 47. Let the arch be convex towards the baſe, and let the ordinate Pp increaſe while the baſe AP increaſes. Let BD and DG be equal lines deſcribed by P with an uniform motion in any equal times that immediately ſucceed after each other. Let the ordinates BC, DE and GH meet the curve in the points C, E and H; let the the tangent at E meet the ordinates GH and BC in T and t; and let MEI parallel to the baſe meet theſe ordinates in I and M. Then, becauſe BD and DG are equal, IT is equal to Mt; and, the arch CEH being convex towards the baſe, IH is greater than IT, or Mt; which is itſelf greater than MC, for the ſame reaſon. Therefore IH is ſurely greater than MC. But MC and IH are the increments of the ordinate that are generated in equal times while the baſe acquires the augments BD and DG; and, ſince thoſe increments continually increaſe, it follows, that the motion with which the ordinate flows is continually accelerated. When the ordinate decreaſes, the decrements IH, MC continually decreaſe, and the ordinate flows with a motion that is continually retarded.

FIG. 50. 185. Let the arch be concave towards the baſe; and, the conſtruction being the ſame, MC ſhall exceed Mt, or IT, which is itſelf greater than IH: ſo that MC being greater than IH, the ordinate flows with a motion that is retarded or accelerated continually according as it increaſes or decreaſes while the baſe increaſes.

FIG. 47. 186. The motion of the point p, which is ſuppoſed to deſcribe the curve, is perpetually accelerated, when, the baſe being ſuppoſed to flow uniformly, the curve is convex towards the baſe, and the ordinates increaſe while the curve increaſes. For the arch EH being greater than its chord EH, which in this caſe exceeds the tangent ET, (becauſe the point T is betwixt I and H, and the angle ETH is greater than EHT,) the arch EH is ſurely greater than the tangent ET. But the arch EC is leſs than the tangent Et, or ET; for if CV parallel to the baſe meet the tangent Et in V, VC ſhall be leſs than Vt; and, the arch EC being leſs than the ſum of EV and VC, (by art. 183.) it muſt be leſs than Et. Therefore the arch EH is greater than CE, and the motion of p in deſcribing the curve is perpetually accelerated. In the ſame manner it appears, that, when

when the curve is convex towards the base, and the ordinates decrease while the curve increases, the motion of p (or that with which the curve Fp flows) is retarded perpetually.

187. But when the curve is concave towards the base, and, the base being supposed to flow uniformly, the curve increases while the ordinates increase, the arch EC is greater than the chord EC, and therefore greater than the tangent Et, or ET, which is itself greater than the sum of EV and VH, (VH parallel to the base being supposed to meet the tangent in V,) and consequently is greater than the arch EH, (by art. 183.) Therefore, in this case, the lines described by p in equal succeeding times perpetually decrease, and its motion (or the motion with which the curve Fp flows) is perpetually retarded. But if the ordinates decrease, in this case, while the curve increases, the motion of p is perpetually accelerated. FIG. 50.

PROP. XIV.

188. *Let* ET *be the tangent of the curve* FE *at* E; *and,* EI *being parallel to the base* AD, *let* IT *be parallel to the ordinate* DE: *Then the fluxions of the base, ordinate and curve, shall be measured by the right lines* EI, IT *and* ET, *respectively.*

First, let the arch CEH be convex towards the base; and, the base being supposed to flow uniformly, the ordinate shall increase with an accelerated motion, by the last lemma. The figure and construction being the same as in the 184th article, it follows from the first and second axioms, that, in the same time that the base acquires the augment DG, a line less than IH, but greater than MC, would be generated by the motion with which the ordinate DE flows continued uniformly. Therefore, if this line be not equal to IT, first let it be equal to some line IL, less than IH, but greater than IT. Join EL, and it shall meet the curve EH in some point R betwixt E and H; let the ordinate RQ meet the tangent in N, the base in Q, and the right line EI in K. Then, since IL is to KR as EI is FIG. 46.

is to ER, or as DG is to DQ, and, when the generating motion is uniform, the quantities generated are in the ſame proportion as the times; it follows, that, if the motion with which the ordinate DE flows was continued uniformly, the line KR would be generated by it in the ſame time that the baſe acquires the augment DQ. But, becauſe the ſame line KR is generated by the accelerated motion with which the ordinate flows, in the ſame time that the baſe by flowing uniformly acquires the augment DQ, it follows from the firſt axiom, that a leſs line than KR would be generated in that time by the motion with which the ordinate DE flows continued uniformly. And theſe being contradictory, it follows, that the line that would be generated by the motion with which the ordinate DE flows continued uniformly while the baſe acquires the augment DG, is not greater than IT. If it be ſaid to be leſs than IT, or M*t*, let it be equal to M*l*; and, M*l* being greater than MC, but leſs than M*t*, it follows, that E*l* muſt meet the arch CE in ſome point *r* betwixt E and C. Let the ordinate *rq* meet the tangent in *n*, the baſe in *q*, and the right line EM in *k*. Then, ſince M*l* is to *kr* as EM is to E*k*, or as DB is to D*q*; it follows, that, if the motion with which the ordinate DE flows was continued uniformly, it would generate a line equal to *kr* in the time that the baſe acquires an augment equal to D*q*. But, ſince the increment *kr* is generated in an equal time by the accelerated motion with which the ordinate flows before *p* comes to E, that is, before the motion with which the ordinate DE flows is acquired; it follows from the ſecond axiom, that a line greater than *kr* would be generated in the ſame time by the motion with which DE flows continued uniformly. And theſe being alſo contradictory, it follows, that IT meaſures accurately the motion with which the ordinate DE flows, or its fluxion, when the motion with which the baſe flows, or its fluxion, is meaſured by DG or EI.

189. In the ſame caſe, the motion of the point *p*, that is ſuppoſed to deſcribe the curve, is perpetually accelerated by art. 186. If the motion of *p* was continued uniformly from E, a line leſs than the arch EH, but greater than the arch CE, would be deſcribed by it in the time P deſcribes DG, by the firſt and ſecond

fecond axioms. If this line be not equal to the tangent ET, firft let it exceed ET by TL; and, becaufe this line is lefs than the arch EH which is lefs than the fum of ET and TH, (by art. 183.) it follows, that TL is lefs than TH, and that the right light line EL muft meet the arch EH in fome point R betwixt E and H. Let the ordinate RQ meet the bafe in Q and the tangent in N. Then, fince ET is to EN, and TL to NR, as DG is to DQ; it follows, that the fum of ET and TL is to the fum of EN and NR as DG is to DQ; and that a line equal to the fum of EN and NR would be defcribed by *p*, if its motion was continued uniformly from the term when it comes to E, in the time P defcribes DQ. But, while the motion of P is uniform, the motion of *p*, in defcribing the arch ER, is perpetually accelerated, (by art. 186.) and it follows from the firft axiom, that, in the time P defcribes DQ, a line lefs than the arch ER (and confequently lefs than the fum of EN and NR) would be defcribed by *p* if its motion was continued uniformly from E. And thefe being contradictory, it follows, that the line which would be defcribed by *p* if its motion was continued uniformly from E, in the time P defcribes DG, is not greater than ET. If this line be faid to be lefs than ET, or E*t*, let E*t* exceed it by *tl*; and, fince that line is greater than the arch EC (which is greater than the chord EC, and therefore is greater than the excefs of E*t* above *t*C,) it follows, that *tl* muft be lefs than *t*C, and that the right line E*l* muft meet the arch EC in fome point *r* between E and C. Let the ordinate *rq* meet the bafe in *q*, and the tangent in *n*: and, fince E*t* is to E*n*, and *tl* is to *nr*, as DB is to D*q*; it follows, that, if the motion of *p* was continued uniformly from E, a line equal to the difference of E*n* and *nr* would be defcribed by it in the time P defcribes a line equal to *q*D. But, the motion of *p* being perpetually accelerated, the fame line muft be greater than the arch *r*E, (by ax. 2.) and therefore greater than the difference of E*n* and *nr*, which is lefs than E*r*, by art. 183. And thefe being alfo contradictory, it appears, that the motion of *p* at the term when it comes to E, or the fluxion of the curve FE, is meafured accurately by ET when the fluxion of the bafe is meafured by DG, or EI.

190. When

FIG. 50. 190. When the curve is concave towards the base, the proposition is demonstrated in the same manner from the third and fourth axioms: or it may be deduced from the preceeding case, by drawing any right line *ao* parallel to the base in such a manner that the arch CEH may have its convexity towards it. For, if DE produced meet *ao* in *d*, and the curve meet *ao* in *f*, the motions with which *d*E and *f*E decrease are equal to those with which DE and FE increase. When the base flows with a variable motion, the proposition is demonstrated from what we have shewn by the eleventh general theorem, and the 60th article.

FIG. 47. 191. COR. In the first case, when the curve is convex towards the base, the part TH of the increment IH of the ordinate DE that is generated while the base acquires the augment DG, is owing to the acceleration of the motion with which the ordinate flows during that time. When the curve is concave FIG. 50. towards the base, the increment IH of the ordinate is less than IT; and the difference arises from the retardation of the motion with which the ordinate flows. The fluxion of the ordinate is the same in all curves that have the same tangent at E.

FIG. 51. 192. The point S, the right line AO, and the circle FDB described about the center S, being given in position, let the point P describe the right line AO, SP meet the circle FDB in N; and let the point M describe any right line SD so that SM may be always equal to SP. Let SA be perpendicular on AO; and, if the motion of the point P from A towards O be uniform, the motion of the point M in the right line SD shall be perpetually accelerated; but the motion of the point N shall be perpetually retarded. For let *h*E, EH be equal lines described by P in any equal times. Let S*h*, SE and SH meet the circle FDB in *g*, D and G. From S as center, through *h* and H, describe the arches *hk*, HK meeting SE in *k* and K, and let *hl*, HL be perpendicular on SE in *l* and L. Then shall *k*E and EK be the lines described by M, and *g*D, DG the arches described by N in the same equal times. Because *h*E is equal to EH, therefore (Elem. 26. 1.) E*l* is equal to EL; and, EK being greater than EL, or E*l*, which is greater than E*k*, it is evident that EK is greater than E*k*; and therefore the motion of M is pepetually accelerated. Because HL is equal to *hl*, but SH

SH greater than Sb, the angle HSL is less than bSL, and the motion of N is perpetually retarded. It is obvious, that, when the motion of M from S towards N is uniform, the motions of P and N are retarded; and that, when the motion of N from f towards b is uniform, the motions of P and M are perpetually accelerated.

PROP. XV.

193. *The point* S *and the right line* AO *being given in position, let the circle* fNb *described from the center* S *meet* AO *in* E; *and the fluxions of the right lines* AE, SE *and of the arch* fE *shall be to each other in the same ratio as the right lines* SE, AE *and* SA. FIG. 52.

While the point P describes AO, let SP meet the circle fEb in N, and upon SE let SM be always equal to SP, as in the preceeding article. Then the fluxions of the right lines AE, SE and of the arch fE shall be measured by the velocities of the points P, M and N at the term when they come together to E. Let the motion of P be uniform, and the motion of M shall be accelerated, by the last article. It is manifest, that the velocity of P is greater than the velocity of M. If the velocity of P be to the velocity of M, at the term when they come to E, in a less ratio than that of SE to AE, let their ratio be the same as that of SH to AH, or (EK being perpendicular on SH in K) that of EH to KH; and the point M shall describe a line equal to KH, if its motion be continued uniformly from E, in the time P describes EH; and, KH being greater than GH the excess of SH above SE, it follows, that a greater line shall be described in the same time by M, when its motion is continued uniformly from E, than when it is continually accelerated from the same term, against the first axiom. If the velocity of P be to the velocity of M at the term when they come to E in a greater ratio than that of SE to AE, let their ratio be the same as that of SC to AC, or (Ek being perpendicular on SCg in k)

 that

that of EC to C*k*; and the point M shall describe a line equal to C*k* by its motion continued uniformly from E in the same time P describes a line equal to CE; and, C*k* being less than C*g* the excess of SE above SC, it follows, that the line described by an accelerated motion is greater than the line which is described in an equal time by the motion acquired by this acceleration continued uniformly, against the second axiom. Therefore the velocity of P is to the velocity of M, at the term when they come to E, as SE is to AE. This might have been demonstrated from the 96th article in a different manner, because the fluxion of the square of SE is equal to the fluxion of the square of AE.

194. The motion of P from A towards O being uniform, the motion of N is perpetually retarded, by the 192d article. If the velocity of P be to the velocity of N as SH greater than SE is to SA, or as EH is to EK, the point N shall describe EK, by its motion continued uniformly from E, in the time P describes EH; and, EK being less than the arch EG which is described in the same time by N with a retarded motion, it follows, that a less line would be described in the same time by a motion continued uniformly, than by the same motion perpetually retarded from the same term, against the third axiom. If the velocity of P be to the velocity of N in a less ratio than that of SE to SA, or (CR being perpendicular in R on ET the tangent of the circle at E) that of CE to ER, let it be the same as that of CE to E*x*; join C*x*, and let S*z* parallel to it meet AE in Q, the arch E*g* in *u*, and the tangent ET in *z*. Then, since E*x* is to E*z* as EC is to EQ, it follows, that the point N would describe a line equal to E*z*, by its motion continued uniformly from E, in the time P describes a line equal to EQ. But the point N describes the arch *u*E by its retarded motion in that time before it comes to E; and, the arch *u*E being less than its tangent E*z*, it follows, that the line described by N while its motion is retarded, is less than the line which would be described in an equal time by the motion that remains after that retardation continued uniformly, against the fourth axiom. Therefore the velocities of the points P, M and N, at the term when they come together to E, are in the same ratio as the right

lines

lines SE, AE and SA; and the fluxions of the right lines AE, SE and of the arch *f*E are in the same ratio. This may be shewn in like manner, by supposing the motion of M or that of N to be uniform; and the proposition is made general by the eleventh theorem.

195. Cor. I. Let a circle A*n*D described from the center S meet SP in *n*, and SE in D; and the velocity of *n* shall be always to the velocity of N as S*n* or SA is to SN or SE, by theor. 3. The velocity of N is to the velocity of P when they come to E in the same ratio. Therefore the velocity of *n* at D is to the velocity of P at E in the duplicate ratio of SA to SE; and the fluxion of the arch AD is to the fluxion of its tangent AE in the same ratio. In the same manner it appears, that the fluxion of the arch AD is to the fluxion of its secant SE as the square of SA is to the rectangle SEA.

196. Cor. II. If the points P and *p* set out with equal velocities from A and *a* in the right lines AO, *ao* at the same time; and, the motion of *p* being continued uniform, the velocity of P be always as its distance from S; then *ap* shall be the logarithm of the sum of SP and AP. For the fluxion of *ap* shall be to the fluxion of AP as SA is to SP, and to the fluxion of SP as SA is to AP. Therefore the fluxion of *ap* shall be to the sum of the fluxions of SP and AP as SA is to the sum of SP and AP; and, when *ap* increases uniformly, the sum of SP and AP increases proportionally, (by art. 160.) Therefore *ap* is the logarithm of the sum of SP and AP, or is the measure of the ratio of that sum to SA, the modulus being SA. The velocity of P is to the velocity of *p* as the velocity of *p* is to the velocity of *n*; and, when P comes to E, the velocity of *p* is equal to the velocity of N.

197. Cor. III. Let the right line *ao* given in position meet AO in F, and SP always intersect *ao* in *p*; then the velocity of P shall be to the velocity of *p* as the rectangle SPF is to the rectangle S*p*F; and, if S*a* be perpendicular on *ao*, the fluxion of AP shall be to the fluxion of *ap* in the same ratio. This may be deduced from the 15th proposition, or immediately from the axioms, thus; If *ao* meet SA in some point betwixt S and A, the motion of *p* shall be accelerated when the motion FIG. 53.

 of

of P from A towards O is uniform. For, while P describes any equal lines CE, EH, let *p* describe *ce* and *eh*; let *le*L parallel to AF meet SH and SC in L and *l*; and it is manifest, that L*e* shall be equal to *el*, but that *eh* shall be greater than *ec*; and therefore the motion of *p* is accelerated. Let L*k* parallel to SE meet *eh* in *k*; and, if the motion of *p* was to be continued uniformly from *e*, the line *ek* would be described by it in the same time P describes EH. For, if it should be said, that it would describe *er* greater than *ek* in that time, let S*q* parallel to L*r* meet EH in Q, *e*L in *x*, and *eh* in *q*. Then, since *er* is to *eq* as *e*L is to *ex*, or as EH is to EQ, it follows, that the point *p*, by its motion continued uniformly from *e*, would describe *eq* in the time P describes EQ. But the point *p* describes the same line *eq* in the same time when its motion is continually accelerated from that term; so that the same space would be described in the same time by *p* when its motion is continued uniformly from *e*, and when it is continually accelerated, against the first axiom. In like manner, it is shewn from the second axiom, that the line which would be described by the motion of *p* continued uniformly from *e*, in the time P describes EH, is not less than *ek*. Therefore that line is equal to *ek*; and the velocity of *p* is to the velocity of P, when they come to *e* and E, as *ek* is to EH; and the fluxion of *ae* is to the fluxion of AE in the same ratio. But the ratio of *ek* to EH is compounded of the ratio of *ek* to *e*L, or that of *e*F to EF, and of the ratio of *e*L to EH, or that of S*e* to SE. Therefore the fluxion of *ae* is to the fluxion of AE as the rectangle S*e*F is to the rectangle SEF. When the motion of *p* is retarded while that of P is uniform, the demonstration is deduced in the same manner from the third and fourth axioms: or the same demonstration may serve, by supposing the motion of *p* in those cases uniform; for the motion of P shall then be accelerated. When the points E and *e* come to F, *ek* is to EH as SA is to S*a*; and the fluxion of *a*F is to the fluxion of AF in that ratio. When *ao* is parallel to AO, it is manifest, that the fluxion of *ae* is to the fluxion of AE as S*e* is to SE, which in this case is an invariable ratio, and is the same as that of S*a* to SA.

198. Cor. IV. The fluxion of SE is to the fluxion of AE as

as AE is to SE; and the fluxion of *ae* is to the fluxion of S*e* as S*e* is to *ae*. Therefore the fluxion of SE is to the fluxion of S*e* as the rectangle AEF is to the rectangle *ae*F. Let S*u* constitute the angle *a*S*u* equal to ASF, and meet *a*F in *u*; then the fluxion of SE shall be to the fluxion of S*e*, when the points E and *e* come to F, as the rectangle contained by AF and S*a* is to the rectangle contained by *a*F and SA, or as *au* is to *a*F.

199. ARCHIMEDES demonstrates, in the 6th proposition of his treatise concerning spiral lines, that a right line SP may be drawn from S meeting AE in P, and the circle *f*E in N, so that PN may be to the chord EN in any ratio less than that of AE to SA. For, if SI parallel to AE meet the chord NE and tangent ZE produced in V and I, PN shall be to EN as SN or SE is to NV; and, since NV may be equal to any right line greater than EI, it follows, that PN may be to EN in any ratio less than that of SE to EI, or that of AE to SA. It is also manifest, that as N approaches to E, NV decreases, and that the ratio of SE to NV or of PN to EN continually decreases. In like manner he shews, in the 7th proposition, that a right line S*p* may be drawn from S meeting the right line AE produced beyond E in *p*, and the circle in *n*, so that *pn* may be to the chord E*n* in any ratio greater than that of AE to SA, or that of SE to EI; because, if E*n* produced meet SI in *v*, *pn* shall be to E*n* as S*n* or SE is to *nv*, and *nv* may be equal to any line less than EI. In this case, when *n* approaches to E, *nv* increases, and the ratio of *pn* to E*n* or of SE to *nv* continually decreases. FIG. 54.

200. In the 8th proposition of the same treatise, he shews, that PN may be to the tangent EZ in any ratio less than that of AE to SA, as that of SE to EL, EL being greater than EI. For let a circle described through L, S and I meet SE produced in R: then, because EL is greater than EI, a right line SZY may be drawn from S meeting this circle in Y, so that ZY may be equal to ER. Supposing that SZ is such a right line, the rectangle SZY being equal to LZI, and the rectangle contained by SZ and EI being equal to the rectangle contained by SP and ZI, because SZ is to SP as ZI is to EI, it follows, that the rectangle LZI is to the rectangle contained by SP and ZI as the rectangle FIG. 55.

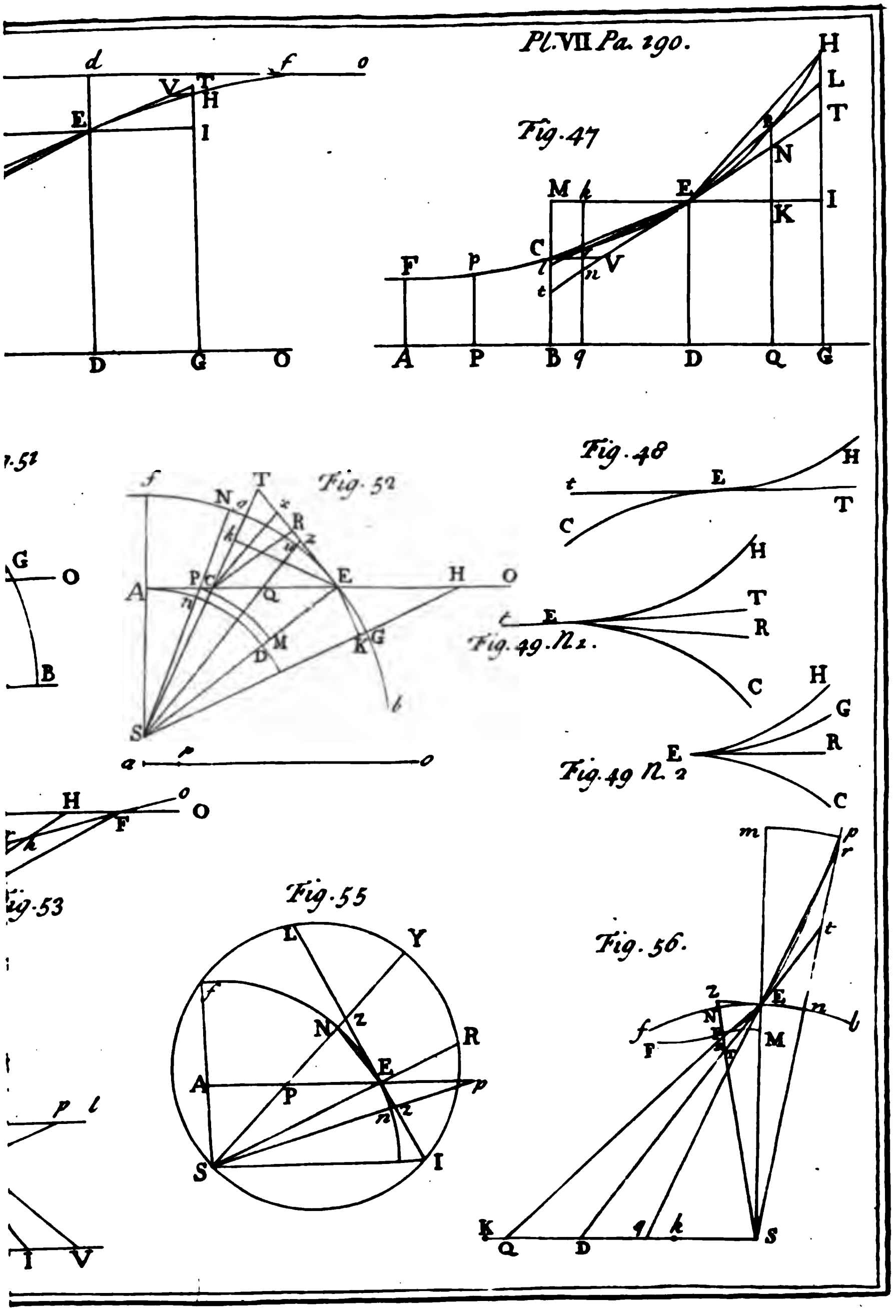
rectangle SZY is to the rectangle contained by SZ and EI. Therefore LZ is to SP as ZY or ER is to EI, or as EL is to ES or NS; and EZ is to PN as EL is to SE. It is easy to see, that when N approaches to E, the ratio of PN to the tangent EZ increases continually. In like manner he shews, in the 9th proposition, that *pn* may be to the tangent E*z* in any ratio that is greater than that of SE to EI, or of AE to SA; and it is easy to shew, that, when *n* approaches to E, the ratio of *pn* to E*z* decreases continually.

FIG. 56. 201. Let P describe an arch FPH of a continued curvature; let SP meet the circle *f*E*b* in N; and suppose that SM is always taken equal to SP, as in art. 192. Then, if SP increase continually, and, the arch being supposed to have its convexity towards S, if the motion of N be uniform, the motion of M shall be perpetually accelerated. For, if the arch EN be equal to E*n*, and the right lines SN, S*n* meet the curve in P and *p*, and its tangent at E in T and *t*, E*t* shall be greater than ET, and the excess of S*p* above SE being greater than the excess of S*t* above SE, which is greater than the excess of SE above ST, it must be greater than the excess of SE above SP. In this case, the motion of P in the curve is also accelerated; for it is manifest, that the arch E*p* is greater than the arch EP. When the arch FPH is concave towards S, and the motion of N is uniform, the motion of M may be uniform, accelerated or retarded.

PROP. XVI.

202. *Let the circle f*E*b meet the curve* FPH *in* E, *and* SD *perpendicular on* SE *meet the tangent* ET
FIG. 57. *in* D; *then the fluxion of the ray* SE *shall be to the fluxion of the arch f*E *as* SE *is to* SD.

Case 1. The motion of N in the arch *f*E*b* being uniform, let the motion of M in the line SE be also uniform; in which case P describes the spiral of ARCHIMEDES. The fluxion of SM is to the fluxion of *f*N as the constant velocity of M is to the constant velocity of N, or as EM is to the arch EN. If the ratio


Pl. VII Pa. 190.
Fig. 47
Fig. 48
Fig. 49 N. 1.
Fig. 49 N. 2
Fig. 52
Fig. 53
Fig. 55
Fig. 56.

tio of those fluxions be greater than that of SE to SD, let it be the same as that of SE to S*k*; join E*k*, and, since ED is the tangent at E, a part of E*k* must fall within the spiral. Let P be a point in the spiral within the angle *k*ED, and let PL perpendicular on SE meet E*k* in V. Then, since the fluxion of SE is to the fluxion of *f*E as SE is to S*k*, or as EL is to VL, EM shall be to the arch EN as EL is to VL. But EM is less than EL, and the arch EN is greater than VL; therefore the ratio of EM to EN is less than the ratio of EL to VL. And these being contradictory, it follows, that the ratio of the fluxion of SE to the fluxion of *f*E is not greater than that of SE to SD. If the ratio of those fluxions be less than that of SE to SD, let it be the same as that of SE to SK; produce DE and KE to *d* and I; and from *p* a point of the spiral within the angle *d*EI, let *pl* perpendicular on SE in *l* meet the right line EI in *u*; let S*p* meet the circle *f*E*b* in *n*, and S*m* be equal to S*p*. Then, since the fluxion of SE is to the fluxion of *f*E as SE is to SK, or as E*l* is to *ul*, E*m* is to the arch E*n* as E*l* is to *lu*. But E*m* is greater than E*l*; E*n* is less than its tangent E*x*, and therefore is less than *lu*; so that E*m* is to E*n* in a greater ratio than E*l* is to *lu*. And these being contradictory, it follows, that the fluxion of SE is to the fluxion of the arch *f*E as SE is to SD. From this it follows, conversely, that if SE be to SD as the velocity of M in the ray SE is to the velocity of N in the circle *f*E*b*, or (supposing S*f* to be the position of the ray SN when P sets out from S the beginning of the spiral) as the ray SE is to the arch *f*E, then shall ED be the tangent of the spiral at E; and this coincides with the 21st proposition of the treatise of ARCHIMEDES concerning that line.

203. *Case* 2. Suppose now the motion of M in the ray SE to be continually retarded while the motion of N in the arch NE*n* is uniform. It follows from the fourth axiom, that, if the motion of M be continued uniformly from the term when it comes to E, a less line than EM will be described by it in the same time N describes an arch equal to EN; and, the fluxions of SE and *f*E being measured by the velocities of M and N at the term when they come to E, it follows, that EM is to EN in a greater ratio than that of the fluxion of SE to the fluxion of

*f*E.

*f*E. If the ratio of thoſe fluxions be greater than that of SE to SD, let it be the ſame as that of SE to S*k*, or that of EL to LV; and, P being a point of the curve within the angle *k*ED, as in the laſt article, EM ſhall be to EN in a greater ratio than EL is to LV; which is impoſſible, becauſe EM is leſs than EL, and EN is greater than LV. If the fluxion of SE be to the fluxion of *f*E in a leſs ratio than that of SE to SD, let their ratio be that of SE to SK, or (the conſtruction being the ſame as in the laſt article) that of E*l* to *lu*. It follows from the third axiom, that E*m* is to E*n* in this caſe in a leſs ratio than that of the velocity of M to the velocity of N at the term when theſe points come to E, or the ratio of the fluxion of SE to the fluxion of *f*E. Therefore E*m* is to E*n* in a leſs ratio than E*l* is to *lu*. But this is impoſſible, becauſe E*m* is greater than E*l*, and E*n* is leſs than *lu*. Therefore the fluxion of SE is to the fluxion of *f*E as SE is to SD.

FIG. 58. 204. *Caſe* 3. Let the motion of M in the ray SE be accelerated continually while the motion of N in the arch NE*n* is uniform, but ſo that the arch PE*p* may be ſtill on the ſame ſide of the tangent ED with S; and the fluxion of SE ſhall be to the fluxion of *f*E in a greater ratio than that of EM to the arch EN, by the ſecond axiom, but in a leſs ratio than that of E*m* to E*n*, by the firſt axiom. Let S*a* be perpendicular on the tangent in *a*; and, if the ratio of thoſe fluxions be that of SE to SK, which is leſs than the ratio of E*a* to S*a*, it follows from the 200dth article, that a right line ST may be drawn from S meeting the tangent ED in T, the circle *f*E*b* in N, and the tangent of the arch EN in Z, ſo that TN may be to EZ as SE is to SK. Let this right line ST meet the curve FE in P; and, ſince TN is to the arch EN (which is leſs than the tangent EZ) in a greater ratio than SE is to SK, and therefore in a greater ratio than EM is to EN, it follows, that TN is greater than EM or PN, and that the point T is between P and S, againſt the ſuppoſition. If the fluxion of SE be to the fluxion of *f*E as SE is to S*k*, then, by the 199th article, a line S*t* may be drawn from S meeting the tangent DE produced beyond E in *t*, and the circle *f*E*b* in *n*, ſo that the ratio of *tn* to the chord E*n* may be the ſame as that of SE to S*k*, or of the fluxion of SE

to

to the fluxion of fE, which is less than the ratio of pn to the arch En. Therefore, supposing St to be such a line, the ratio of tn to the arch En shall be less than the ratio of pn to the same arch; and tn shall be less than pn, so that the point t shall fall betwixt p and S, against the supposition. Therefore the fluxion of SE is to the fluxion of fE in a ratio that is neither greater nor less than that of SE to SD.

205. *Case* 4. Let the point S and the curve FPp be on different sides of the tangent ED; and, the motion of N being uniform, the motion of M must be perpetually accelerated, by the 201st article. In this case, the fluxion of SE is to the fluxion of fE in a greater ratio than that of EM to the arch EN, by the second axiom; but in a less ratio than that of Em to the arch En, by the first axiom. If the ratio of these fluxions be that of SE to any line SK greater than SD, let Q be any point betwixt D and K; join EQ; and, by the 200dth article, a right line SR may be drawn from S to the right line EQ meeting it in R, the circle fEb in N, and its tangent in Z, so that RN shall be to EZ as SE is to SK, and, consequently, in a greater ratio than that of EM to the arch EN. Suppose therefore SR to be such a right line, and RN shall be greater than EM or PN; and, because the ratio of RN to EZ continually increases while N moves towards E, it follows, that while N describes the arch NE, the ratio of RN to the arch EN is greater than that of SE to SK, and that RN is greater than PN. But RN is less than TN; therefore the right line RE passes through the angle of contact PET formed by the curve PE and its tangent ET; which is absurd, by art. 181. If the fluxion of SE was to the fluxion of fE as SE is to any line Sk less than SD; then, taking any point q betwixt D and k, joining qE, and producing it beyond E, a right line Sr might be drawn to it (by art. 199.) meeting it in r, and the circle Eb in n, so that rn might be to the chord En as SE is to Sk, or as the fluxion of SE is to the fluxion of fE, and, consequently, in a less ratio than that of Em, or pn, to the arch En. Let Sr be such a line, and rn shall be less than pn. Suppose the point n to move from n towards E, and the ratio of rn to En shall decrease continually by art. 199. and shall be less than that of SE to FIG. 56.

 Sk;

Sk; therefore, during that time, rs is less than pn; but it is greater than tn; so that the right line Er must pass through the angle of contact pEt formed by the curve Ep and its tangent Et; which is absurd, by art. 181. Therefore the fluxion of SE is to the fluxion of fE precisely as SE is to SD. By joining these cases together, the demonstration is applicable when the motion of M is accelerated on one side of E and retarded on the other; and by the ninth and eleventh theorems, and the 60th article, it is rendred general. When the tangent ET is perpendicular to SE, and the velocity of N is given, the velocity of M at E and the fluxion of SE vanishes. When the ray SE touches the curve in E, and the velocity of M in the ray SE is given, the velocity of N at E and the fluxion of fE vanishes. This proposition may be demonstrated in another manner; but this seems to have the nearest affinity to the method of ARCHIMEDES.

PROP. XVII.

206. *The same things being supposed as in the last proposition, the fluxion of the curve* FE *is to the fluxion of the ray* SE *as the tangent* ED *is to the ray* SE. FIG. 59.

For let SP meet the tangent always in T, and the velocities of the points P and T shall become equal at the term when they come together to E.

First, let the curve FPp be convex towards S, and SP increase continually, as in the last article; then, the motion of N being supposed uniform, the motion of P in the curve shall be accelerated perpetually, by art. 201. The motion of the point T is also accelerated, and its velocity at the term when it comes to E is less than the velocity of P at any term after P passes E, as when it comes to p; because a less line than Et would be described by the motion of T continued uniformly from E, in the same time that P would describe a greater line than the arch Ep (which exceeds Et) by its motion continued uniformly from

p,

p, by the first and second axioms. Therefore, if the velocity of T at the term when it comes to E was greater than the velocity of P at that term, it might be equal to the velocity of P at the term when it comes to some point b betwixt E and p. Let Sb meet Et in g; and it follows from the second axiom, that the point P would describe a line greater than the arch Eb by its motion continued uniformly from b, in the time it describes Eb with its accelerated motion; so that the point T would describe a greater line than Eg by its motion continued uniformly from E in the same time that it describes Eg with its accelerated motion, against the first axiom. In like manner it appears, that the velocity of T at the term when it comes to E cannot be less than the velocity of P at the same term. Therefore these velocities are equal, and, by prop. 15. the fluxion of the curve FP is to the fluxion of the ray SE as ED is to SE.

207. When the point P describes a circle that passes through the point S, the motion of N being supposed uniform, the motion of P is also uniform. In this case, PE is always greater than TE, and pE less than tE. Therefore the velocities of P and T are equal at the term when they come to E, by the 53d article. When the point P describes any other curve that is concave towards S, its motion is in some cases accelerated and in others retarded. But it follows from what was shewn in the 54th, 55th and 56th articles, that the velocities of P and T are equal at the term when they come to E in those cases also. Therefore, in general, the fluxion of the curve FP is to the fluxion of the ray SE as ED is to SE.

208. Angles are measured by the arches which subtend them in equal circles; and in general they are to each other in the ratio compounded of the direct ratio of the arches which subtend them in any circles, and the inverse ratio of the rays of these circles. The angular motion of the ray SN that generates any angle ASN is as the velocity of N in the circle EN when the ray SN is given, and is always as the velocity of N directly and the ray SN inversely. The fluxion of the angle ASN is in the same ratio. FIG. 60.

209. Let the right lines CP, SP revolve about the given points C and S; and let their intersection P describe any curve FEH

 in

in the same plane with those points; produce SC till it meet the curve in E, and when P comes to E, the angular motion of CP shall be to the angular motion of SP as SE is to CE. For let the circle EN described from the center S through E meet SP in N; and let the circle ER described from the center C through E meet CP in R; and let SD, CB be perpendicular on AE the tangent of the curve at E in D and B. Then the velocity of N shall be to the velocity of P at the term when they come together to E, as SD is to SE. The velocity of R shall be to the velocity of P at the same term, as CB is to CE; and SD is to SE as CB is to CE. Therefore the velocity of N is equal to the velocity of R at that term; and the angular motion of SP is to the angular motion of CP, when P comes to E, as CE is to SE, by the last article.

PROP. XVIII.

210. *Let* P, *the intersection of the right lines* CP *and* SP *revolving about the given points* C *and* S, *describe the curve* FEH; *let* AO *touch this curve in* E; *constitute the angle* SET *equal to* CEA *so that* ET *and* EA *may ly different ways from* SE *and* CE; *and let* ET *meet* CS *in* T. *Then, if* CA, Sa *be right lines given in position, the fluxion of the angle* ACP *shall be to the fluxion of* aSP, *when* P *comes to* E, *as* ST *is to* CT.

Fig. 61. & 62.

From the centers C and S describe through the point E the circles ER and EN, meeting CP and SP in R and N respectively; and let CB and SD be perpendicular on the tangent AO in B and D. Let CK parallel to ET meet SE, produced if necessary, in K; and let EQ be perpendicular on CK in Q. The angle QKE is equal to SET or CEB by the construction; and therefore the triangle QKE is similar to CEB, so that CE is to CB as EK is to EQ. The triangles SED, CEQ are also similar, and SE is to SD as CE is to EQ. But when P and R come

come to E, the velocity of P is to the velocity of R as CE is to CB, (by prop. 15. & 17.) that is, as EK is to EQ. The velocity of P is to the velocity of N at the same term as SE is to SD, or as CE is to EQ. Therefore the velocity of R is to the velocity of N, when they come together to E, as CE is to EK. This ratio compounded with the inverse ratio of the ray CR to the ray SN, or with the ratio of SE to CE, gives the ratio of SE to EK, or (because ET is parallel to CK) of ST to CT; which therefore (by art. 208.) is the ratio of the angular velocity of CP about C to the angular velocity of SP about S when P comes to E, or the ratio of the fluxion of the angle ASE to the fluxion of the angle *a*SE. When the angle CEA is equal to CSE, the point K falls on S, the angular motions of CE and SE are equal, and the fluxion of the angle ACE is equal to the fluxion of *a*SE. When the tangent AE passes through S, the point T falls on C; and in this case, the angular velocity of the ray CE being given, that of the ray SE vanishes. The point E may coincide with S, but we reserve this case till we come to treat of the curvature of lines. The fluxion of the angle CPS is equal to the sum or difference of the fluxions of the angles ACP, *a*SP, and is to the fluxion of the angle ACP or the fluxion of ASP, when P comes to E, as CS is to ST or to CT. When the points C and S are not in the plane of the curve FEH, the ratio of the angular velocities of the rays CE and SE may be deduced from this proposition. For if V be any point in the tangent AE, and the right line V*s* equal to VS be drawn in the plane CVE constituting the angle *s*VE equal to SVE, the angular velocity of *s*E about *s* shall be equal to the angular velocity of SE about E. FIG. 63.

211. The preceeding propositions shew the analogy there is betwixt the method of Fluxions and the method of Tangents, and serve for determining the tangents from the fluxions of lines and angles, as well as for finding these fluxions from the tangents. Besides these general theorems, there are many particular propositions that are often of use in determining the tangents of curve lines; some of which we shall briefly describe. Let S be a given point in the plane of the curve ALB, LP a tangent at L any point of the curve, SP perpendicular on the FIG. 64.

tan-

tangent in P; and let P be always found in the curve DPE. Let PT constitute with SP an angle SPT equal to the angle SLP, on the same side of SP as LP is of SL; and PT shall be the tangent of the curve DPE at P. For, first, let the arch L*l* of the curve ALB be concave towards S, and the rays drawn from S to the arch L*l* decrease from SL to S*l*, and let S*p* be perpendicular on *lp* the tangent at *l* in *p*; join P*p*, and let *pl* meet PL in R: then, because the angles SPR, S*p*R are right, the angle SP*p* is equal to SR*p*; and the angle TP*p* is equal to the difference betwixt the angles SLP and SR*p*. The angle SLP is equal to the difference of the angles SRP and RSL; the angle SR*p* is equal to the difference of the angles SRP and PR*p* or PS*p*: therefore the angle TP*p* is equal to the difference of the angles PS*p* (or PR*p*) and RSL. But, by supposing *l* to move towards L, the angles PS*p* and RSL and their difference may become less than any given angle. Therefore, while *p* moves towards P, the angle TP*p* may become less than any given angle. From which it follows, that no right line can be drawn through the angle of contact formed by the right line PT and the arch P*p*. Therefore PT is the tangent at P. When ALB is a parabola, and S its focus, DPE is a right line; but, when ALB is any other curve, DPE is a curve line. In the parabola, SR*p* is an angle always of the same magnitude wherever the point *l* be taken, the point L being given; and PS*p* is always equal to RSL or one half of LS*l*. In other cases, according as the angle PS*p* is greater or less than RSL, (or the fluxion of the angle ASP is greater or less than one half of the fluxion of the angle ASL,) the angle SPT is greater or less than SP*p*, and the arch P*p* is concave or convex towards S.

Fig. 65. 212. When the arch L*l* is convex towards S, the construction in other respects being the same, produce *p*P to p; and the angle SPp shall be equal to SR*p*. Therefore the angle TPp is equal to the difference of the angles SR*p* and SLP, which in this case is equal to the sum of PR*p*, or PS*p*, and RSL; and, since this sum may become less than any given angle when *l* moves towards L, it follows, that PT is the tangent of the curve DPE at P. In this case, the arch P*p* is concave towards S. Let ST be perpendicular on PT in T, and let the point T be always

always found in the curve FT; then the tangent of this curve at T shall constitute an angle with ST equal to SPT or SLP. There is a series of curves which may be conceived to be derived from each other in this manner: The tangents form always equal angles with the rays drawn from S at the corresponding points of the curves, and the fluxions of the curves are to the fluxions of the rays at these points in the same ratio in them all.

213. Let any curvilineal figure CEH be the base of any conical surface that has its vertex in V, and ET be the tangent of the arch EH in E. Let *ceb* be a section of this conical figure made by any plane; and let *et* be the common section of that plane and the plane VET: then shall *et* be the tangent of the arch *eb*. For, if *et* meet the arch *eb* in any point besides *e*, it is manifest that ET must meet the arch EH in some other point besides E; and if any right line, as *ex*, can be drawn through the angle of contact *bet* formed by the arch *eb* and the right line *et*, let the common section of the planes V*ex*, CEH be EX; and EX shall pass through the angle of contact HET, against the supposition. It is obvious, that the arch *eb* and its tangent *et* are the shadows of the arch EH and tangent ET formed by rays issuing from V upon the plane *ceb*. Fig. 66.

214. Let the right lines CH, SK revolving about the poles C and S by their intersection P describe the curve APS that passes through S; and let S*k* be the situation of the right line SK when CH passes through S; then shall S*k* be the tangent at S. Fig. 67.

CHAP. VIII.

Of the Fluxions of curve Surfaces.

215. ARCHIMEDES establishes his theorems concerning curve surfaces upon this principle, That, when two curve surfaces have their concavities turned the same way and have the same terms, that which includes the other is the greater surface: and this axiom is sufficient for demonstrating the cases that were considered by him. But, because it cannot be applied

for

for comparing ſurfaces that are generated by curves which are convex towards the axis about which the figure is ſuppoſed to revolve, we ſhall make uſe of the following principle in place of it, that is more general, and ſeems to be no leſs evident. Let CEH be any arch of a curve; and let BG the axis be divided by a continual biſection into any number of equal parts
Fig. 68. BK, KD, DL, LG. Let the ordinates BC, KM, DE, LN, GH
n. 1, 2. meet the curve in the points C, M, E, N, H; and let the tangents at C, M, E, H form the circumſcribed figure CQRSVH. Then, by continuing to biſect the parts of the axis BG, and ſuppoſing the ordinates, chords and tangents to be drawn as before, the perimeters of the circumſcribed and inſcribed figures CQRSVH, CMNEH ſhall approach continually to the arch CEH; and, ſuppoſing the whole figure to revolve about the axis BG, the ſurfaces deſcribed by thoſe perimeters ſhall approach to the ſurface deſcribed by the arch CEH, ſo that the differences betwixt them may become leſs than any given ſurface. The axiom cited from Archimedes in art. 183. may be deduced from this principle.

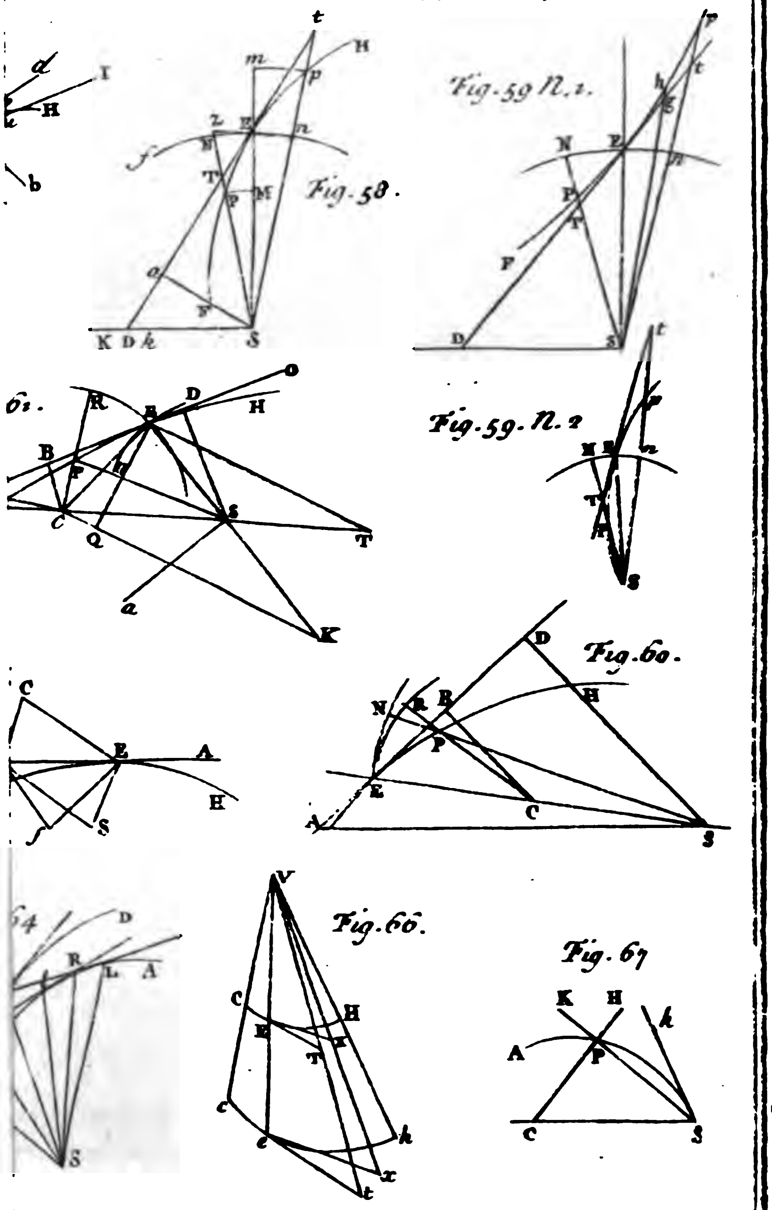
LEMMA VIII.

216. *Let the right line* HC *produced meet the axis* AG *in* A, *and the ſurface deſcribed by* HC *revol-*
Fig. 69. *ving about* AG *ſhall be equal to the area of a circle the radius of which is a mean proportional betwixt* 2HC *and the right line* DE *that biſects* HC *in* E, *and is perpendicular to the axis in* D.

Let HG and CB perpendicular to the axis meet it in G and B. By what was ſhewn after Archimedes, in the Introduction, pag. 11. the ſurface deſcribed by AH is equal to a circle the radius of which is a mean proportional betwixt AH and HG; and the ſurface deſcribed by AC is equal to the area of a circle that has its radius a mean proportional betwixt AC and BC. Therefore theſe ſurfaces are to each other as the rectangle AHG is to the rectangle ACB, or as the ſquare of AH is to the ſquare of

AC;

Pl. VIII. Pa. 200.



AC; and the surface described by CH is to the surface described by AH, as the difference of the squares of AH and AC is to the square of AH, that is, (Elem. 8. 2.) as the rectangle contained by 4AE and EH is to the square of AH, or as the rectangle contained by 2CH and DE is to the rectangle AHG. But the area of a circle whose radius is a mean proportional betwixt 2CH and DE is to the surface described by AH in the same ratio. Therefore the surface described by CH is equal to the area of a circle whose radius is a mean proportional betwixt 2CH and DE. This coincides with the 16th proposition of ARCHIMEDES's treatise concerning the sphere and cylinder.

217. COR. I. When the axis of the cone AB increases uniformly, the convex surface of the cone described by the right line AC increases with a motion that is continually accelerated; for when BG the increment of the axis is given, the surface described by CH (which is the simultaneous increment of the conical surface) is as DE, which increases in the same proportion as AD increases.

218. COR. II. Let the right lines HC, HK meet BC that is perpendicular to the axis BG, in C and K, and let them meet DE in E and L. Let HN bisect the angle CHK, and HI perpendicular to HN shall meet DE produced beyond E when the angle GHN is less than a right one; let them meet in I. Then, the figure being supposed to revolve about the axis BG, the surface described by HC shall be to the surface described by HK as the rectangle DEI is to the rectangle DLI. For let LR perpendicular to HN meet HE in R; and, the angle RLH being equal to LRH, HL shall be equal to HR; and EH shall be to LH (or RH) as EI is to LI; so that the rectangle DEH shall be to the rectangle DLH as DEI is to DLI. But, by this lemma, the surface described by HC is to the surface described by HK as the rectangle DEH is to the rectangle DLH, and, consequently, as the rectangle DEI is to DLI.

219. COR. III. It follows from the last corollary, that, when DE is equal to LI, the surfaces described by HC and HK are equal. Therefore, if GH be produced beyond H till HM be equal to GH, AF parallel to GH meet HK in F, FM meet FIG. 70.

GO perpendicular to HN in O, and BC produced pass through O; the surfaces described by HC and HK shall be equal. For, in this case, DE, HI and OM intersect each other in the point I that is middle betwixt O and M; and, because GH is equal to HM, DE is equal to LI.

Fig. 71. 220. Cor. IV. Let CV be perpendicular on HK in V; produce it till it meet DE in S: and, when ES is either equal to ED, or less than it, the surface described by HC is greater than the surface described by HK. For, in the triangles ECS, EHI, the side EC is equal to EH, the angle CES is equal to HEI, but the angle ECS is greater than EHI, (because the angle ECS is equal to the right angle CVH added to CHV, whereas EHI is only equal to the right angle NHI added to CHN;) and therefore ES is greater than EI. Therefore, when ES is either equal to ED, or less than it, DE is greater than EI, the rectangle DEI is greater than DLI, and the surface described by HC is greater than the surface described by HK.

221. Cor. V. The same things being supposed as in the last corollary, let *k* be any point betwixt C and K, and the surface described by HC shall be greater than the surface described by H*k*, which is itself greater than that which is described by HK. For, let C*u* perpendicular on H*k* in *u* be produced till it meet DE in *s*, and, E*s* being less than ES, (which is supposed to be either equal to ED or less,) it follows, that the surface described by HC is greater than that which is described by H*k*. Let H*k* meet EL in *e*, and *ko* perpendicular on HK meet *e*D in *s*, and *es* being less than *e*D, the surface described by H*k* must be greater than that which is described by HK, by the last corollary.

Fig. 72. 222. Cor. VI. Let *h* be a point on the line HC betwixt H and C, or any where within the triangle CHK; let *hk* parallel to HK meet CK in *k*, and *hc* meet CK in any point *c* betwixt C and *k*; then the surface described by *hc* shall be greater than that which is described by *hk*. For let *hg* parallel to HG meet the axis in *b*, let *de* parallel to *hg* bisect B*b* in *d*, and meet *hc*, *hk* in *e* and *l*; and it is manifest, that if *cu* be perpendicular to *hk*, or parallel to CV, it shall meet *ed* in some point *f* betwixt *e* and *d*.

223. Cor.

223. Cor. VII. It appears in the ſame manner, that, when *b* is any point within the triangle CHK, *c* a point on CK, *k* a point on *c*K, if *cb* and *kb* produced beyond *b* meet CH in *r* and *n*, ſo that the angles C*rb*, C*nb* be each leſs than the angle CHK, then the ſurface deſcribed by *bc* ſhall be greater than the ſurface deſcribed by *bk*. Fig. 73.

224. Cor. VIII. Let HEC be now an arch of a continued curvature convex towards the axis BG; let the ordinates from BC to GH increaſe continually, and let the ordinate DE biſect BG; let HT the tangent at H meet BC in T, and ſuppoſe that T*u* perpendicular on CQ the tangent at C meets DE in *ſ*, ſo that E*ſ* is either equal to the ordinate DE, or leſs than it. Then the ſurface deſcribed by the arch HEC revolving about the axis BG ſhall be leſs than the ſurface deſcribed by the tangent HT revolving about the ſame axis. For let H*k* parallel to the tangent CQ meet BC in *k*, and T*u* in V, the angle TVH being a right one, and the arch HEC being within the triangle TH*k*, it appears, that, if the chord HE be produced till it meet CT in *r*, the ſurface deſcribed by HT ſhall be greater than that which is deſcribed by H*r*, by cor. 5. and the ſurface deſcribed by E*r* greater than that which is deſcribed by EC, by cor. 7. Therefore the ſurface deſcribed by the tangent HT is greater than the ſum of the ſurfaces which are deſcribed by the chords HE and EC. In like manner, the parts of the axis BD, DG being biſected by the ordinates KM, LN, if the chords HN, NE, EM be produced till they meet CT in *u*, *x* and *z*, the ſurfaces deſcribed by N*u*, E*x*, M*z* ſhall exceed the ſurfaces deſcribed by N*x*, E*z*, MC reſpectively, by cor. 7. and therefore the ſurface deſcribed by HT being greater than that which is deſcribed by H*u*, which exceeds that deſcribed by HN*x*; and, this ſurface being greater than that which is deſcribed by HNE*z*, which exceeds the ſurface deſcribed by HNEMC, it follows, that the ſurface deſcribed by HT is greater than that which is deſcribed by HNEMC. In general it appears, that the ſurface deſcribed by the tangent HT is greater than the ſurface which is deſcribed by the perimeter of any rectilineal figure inſcribed in the arch HEC; and it may be ſhewn in the ſame manner, that it is greater than the ſurface deſcribed by the perimeter Fig. 68. n. 1.

rimeter

rimeter of any rectilineal figure circumscribed about the arch formed by its tangents. Therefore the surface described by the tangent HT is greater than that which is described by the arch HEC, by art. 215.

225. Cor. IX. Join GC, and, if the angle GCQ which it forms with the tangent at C be either a right angle, or greater than a right one, and CT be not less than TB, the surface described by the tangent HT shall be greater than that which is described by the arch HEC: for, in this case, E*f* is either equal to ED, or less than it. It appears therefore, that, when the curve is convex towards the axis, and the ordinates increase while the axis increases, an arch of the curve, as CH, may be taken of a finite magnitude so that the surface described by it shall be less than the surface which is described by HT the tangent at H terminated in T by the ordinate from C. If the tangent CQ meet GH in Q, it is evident, that the surface described by CMENH the perimeter of any rectilineal figure inscribed in the arch CEH is always greater than the surface described by the tangent CQ. Therefore the surface described by the arch CEH exceeds that which is described by the tangent CQ, by art. 215.

226. Cor. X. Let the curve FEH be convex towards the axis AG, and the ordinates *pm* increase while the axis A*p* increases; then, if the axis A*p* increase uniformly, the surface described by the curve F*m* shall increase with an accelerated motion. For, while the axis acquires the equal augments BD, DG, let the curve acquire the augments CME and ENH; and it is manifest, that the surface described by the perimeter of any rectilineal figure inscribed in the arch HNE is greater than the surface described by the perimeter of a rectilineal figure inscribed by a similar construction in the arch EMC. Therefore the surface described by the arch EH is greater than that which is described by the arch CE. Let CQ meet DE in *q*, and the tangent at H meet DE in *t*; then, the surface described by the arch CE being greater than that which is described by the tangent C*q*, and the surface described by the arch EH being less than that which is described by the tangent H*t* when CEH is diminished as in cor. 9. it follows, that the motion with

with which the ſurface deſcribed by F*m* flows, while *m* deſcribes the arch CEH, is accelerated in a continued manner.

227. Cor. XI. The reſt remaining as in the laſt corollary, when the arch CEH is concave towards the axis BG, BD may be taken ſo ſmall that the ſurface deſcribed by the tangent ET ſhall be leſs than the ſurface deſcribed by the chord CE, (by cor. 4.) and therefore leſs than the ſurface deſcribed by the arch CE, which exceeds that which is deſcribed by the chord CE, by art. 215. The ſurface deſcribed by the arch EH is leſs than that which is deſcribed by the tangent E*t*, by the ſame article. In this caſe, when the axis increaſes uniformly, the motion with which the ſurface deſcribed by the curve flows may be uniform, accelerated or retarded. Fig. 75.

LEMMA IX.

228. *Let* DE *and* G*t* *perpendicular to the axis* *a*G *meet* *a*E *in* E *and* *t*; *and, in the ſame time that the motion with which the axis* *a*D *flows continued uniformly generates* DG, *let the motion with which the conical ſurface deſcribed by* *a*E *flows continued uniformly generate a ſpace equal to* R; *then ſhall the ſpace* R *be equal to the area of a circle the radius of which is a mean proportional betwixt* DE *and* 2E*t*. Fig. 74.

For, if the ſpace R be greater than ſuch a circle, let A*d* be greater than AD in the ſame ratio; and, if *do* parallel to DE meet E*t* in *o*, R ſhall be equal to a circle whoſe radius is a mean proportional betwixt *do* and 2E*t*; becauſe the area of this circle is to the area of a circle whoſe radius is a mean proportional betwixt DE and 2E*t* as *do* is to DE, or as A*d* is to AD. The ſpace which is generated by the motion with which the ſurface *a*E*e* flows continued uniformly, in the time that the axis *a*D by flowing uniformly acquires the augment D*d*, is to the ſpace R as DG is to D*d*, (by theor. 2.) or as E*t* is to E*o*, and therefore is equal to the area of a circle whoſe radius is a mean proportional betwixt *do* and 2E*o*; but the

the area of this circle is greater than the ſurface deſcribed by the right line E*o*, by lemma 8. and the conical ſurface *a*E*e* increaſes with an accelerated motion when the axis increaſes uniformly, by art. 217. Therefore a greater ſpace is generated in the ſame time by a motion continued uniformly, than when the ſame motion is continually accelerated, againſt the firſt axiom. In the ſame manner it appears from the ſecond axiom, that the ſpace R is not leſs than the area of a circle the radius of which is a mean proportional betwixt DE and 2E*t*.

PROP. XIX.

229. *Let* DE *and* GH *perpendicular to the axis meet the curve in* E *and* H, *and let* GH *meet the tangent at* E *in t; then, the fluxion of the axis being re-* FIG. 74. *preſented by* DG, *the fluxion of the ſurface deſcribed by the curve* FE *ſhall be accurately meaſured by the area of a circle the radius of which is a mean proportional betwixt* DE *and* 2E*t*.

Let the ordinate PM always meet the curve in M and the tangent *a*E in N; and the motions with which the ſurfaces F*fm*M, *a*N*n* flow ſhall be equal at the term when M and N come together to E, or the fluxion of the ſurface F*fe*E ſhall be equal to the fluxion of the ſurface *a*E*e*. For, the conſtruction being the ſame as in the 226th and 228th articles, firſt let the arch CEH be convex towards the axis BG, and the ſurface F*fm*M ſhall flow with an accelerated motion while P and M deſcribe BG and CH, by art. 226. Suppoſe, firſt, BD to be ſo ſmall, according to the 225th article, that the perpendicular from T upon the tangent at C may meet DE in D, or in ſome point betwixt D and E, and the ſurface deſcribed by ET ſhall be greater than the ſurface deſcribed by the arch EC. But the latter ſurface is greater than the ſpace which would be generated by the motion with which the ſurface F*fc*C flows continued uniformly, in the time P deſcribes BD with an uniform motion, (by the firſt axiom;) and the former is leſs than the ſpace

ſpace which would be generated in an equal time by the motion with which the ſurface *a*E*e* flows continued uniformly, by the ſecond axiom. Therefore the motion with which the ſurface *a*E*e* flows, is greater than that with which the ſurface F*fc*C flows; and it is manifeſt from the ſame axioms, that it is leſs than the motion with which the ſurface F*fb*H flows; ſo that it muſt be equal to the motion with which the ſurface F*fm*M flows when M comes to ſome intermediate point of the arch CH. If it be ſaid to be equal to the motion with which the ſurface F*fx*X flows, X being ſome point betwixt E and H, let *d*X perpendicular to the axis in *d* meet the tangent E*t* in *o*. The motion with which the ſurface F*fx*X flows continued uniformly generates a ſpace greater than the ſurface E*ex*X in the time P deſcribes D*d*, by ax. 1. The ſurface E*ex*X is greater than the ſurface deſcribed by the right line E*o* revolving about D*d*, (by art. 225.) Therefore the motion with which the ſurface *a*E*e* flows continued uniformly, would generate a ſpace greater than the ſurface deſcribed by E*o*, in the time P deſcribes D*d* uniformly: But a leſs ſpace than this ſurface would be deſcribed in that time by the motion with which *a*E*e* flows continued uniformly, by the firſt axiom: And theſe are contradictory. Therefore the motion with which the ſurface *a*E*e* flows, is not greater than the motion with which the ſurface F*fe*E flows. Nor can it be leſs. For, if it be ſaid to be equal to the motion with which the ſurface F*fy*Y flows, Y being any point of the curve betwixt E and C, let Yd meet the axis in d, and the tangent ET in o. Then, the ſurface deſcribed by Eo about the axis BD being greater than the ſurface E*ey*Y, (by art. 225.) and this ſurface being greater than the ſpace which would be generated by the motion with which F*fy*Y flows continued uniformly in the time P deſcribes dD uniformly, (by ax. 1.) it follows, that the ſpace which would be generated in that time by the motion with which *a*E*e* flows is leſs than the ſurface deſcribed by Eo: But it is greater than that ſurface by the ſecond axiom: And theſe are contradictory. Therefore the motions with which the ſurfaces F*fe*E, *a*E*e* flow are equal. By the laſt lemma, the fluxion of the axis AD being repreſented by DG, the fluxion of the ſurface *a*E*e* is meaſured by the area of a circle the radius of which

which is a mean proportional betwixt DE and 2E*t*. Let DG be increased in any proportion, and E*t* shall be increased in the same proportion; so that, if DG represent the fluxion of AD, the fluxion of the surface F*fe*E shall be accurately measured by the area of a circle the radius of which is a mean proportional betwixt DE and 2E*t*.

FIG. 75. 230. Let the arch CEH be now concave towards the axis; and BD may be taken in this case so small that the surface described by the arch CE shall be greater than the surface described by the tangent ET, (by art. 227.) but the surface described by E*t* is always greater than that which is described by the arch EH. When the axis increases uniformly, the motion with which the conical surface described by *a*N flows is continually accelerated, by art. 219. and, whether the motion with which the surface F*fm*M flows be uniform, accelerated or retarded, it follows from what was demonstrated in the 53d, 54th and 56th articles, that the motions with which the surfaces F*fe*E, *a*E*e* flow are equal. Therefore, the fluxion of the axis AD being represented by DG, the fluxion of the surface F*fe*E is measured by the area of a circle whose radius is a mean proportional betwixt DE and 2E*t*. The demonstration is rendred general by the propositions which have been so often cited from the first chapter.

FIG. 76. 231. COR. I. Let EO perpendicular to the tangent E*t* meet the axis in O; and, because EO is to DE as E*t* is to DG, it follows, that the fluxion of the surface F*fe*E is measured by the area of a circle whose radius is a mean proportional betwixt 2EO and DG, when the fluxion of the axis is represented by DG. Therefore, when FEH is an arch of a circle whose diameter AB coincides with the axis of motion, the perpendicular EO being invariable, the spherical surface AE*e* flows uniformly when the axis AD flows uniformly; and, in the same time that the uniform motion with which the axis flows generates AD, the uniform motion with which the spherical surface flows generates a surface equal to the area of a circle whose radius is a mean proportional betwixt AD and AB; that is, the spherical surface AE*e* is equal to the area of a circle described with the chord AE; and the whole surface of the sphere is equal to a circle whose radius is AB the diameter of the sphere, and therefore is

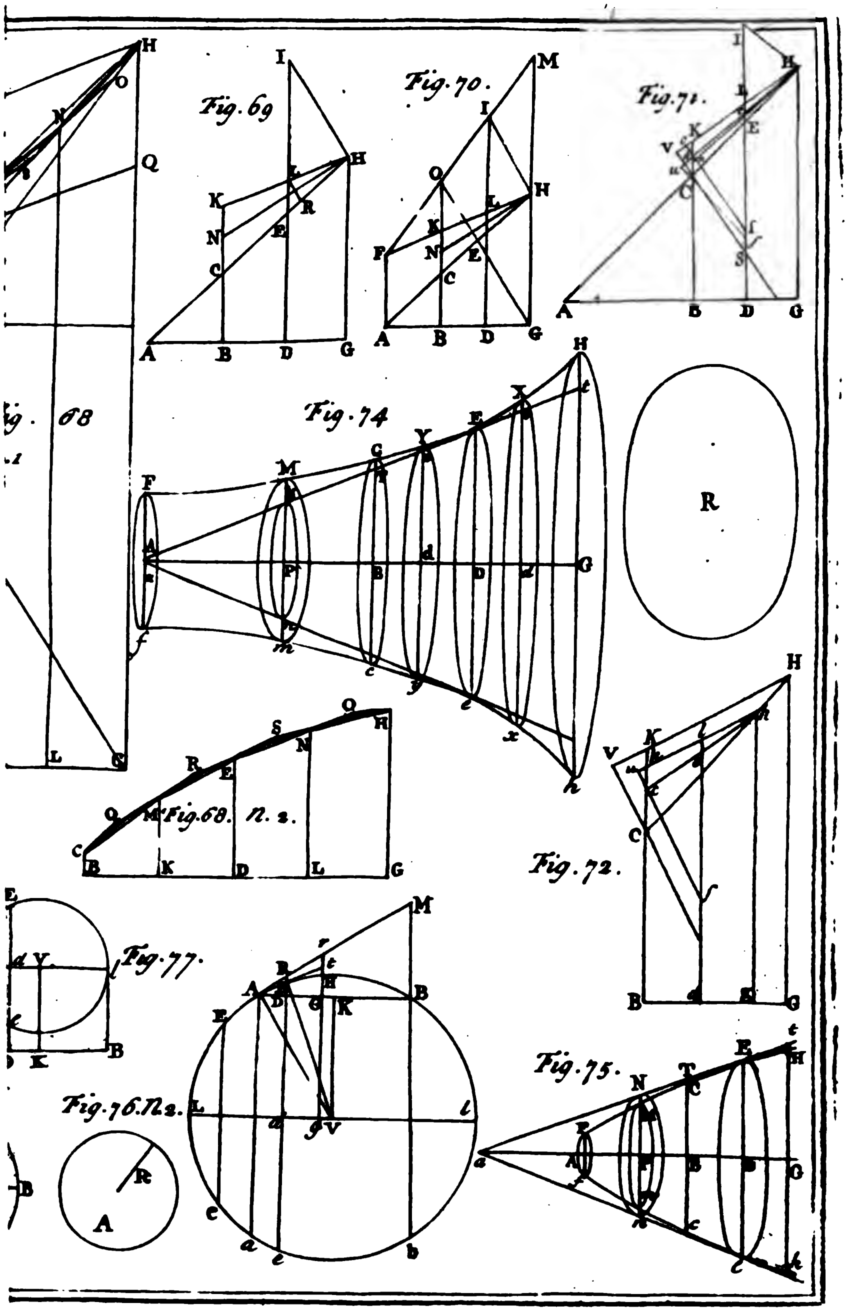
is quadruple of a great circle of the ſphere, as was demonſtrated after ARCHIMEDES in the Introduction.

232. COR. II. But if the circle be ſuppoſed to revolve about a chord AB that is not a diameter, let VK perpendicular from the center V upon AB meet it in K; let L*l* be the diameter parallel to AB, A*a* and B*b* perpendicular to L*l* meet the circumference in *a* and *b*; produce DE till it meet the circumference again in *e*, and let it meet AR the tangent at A in R. Suppoſe the right line R to be the radius of a circle whoſe area is A. Then, if R be a mean proportional betwixt 2VK and the exceſs of AR above the arch AE when E is taken upon the leſſer arch ſubtended by AB, the area A ſhall be equal to the ſurface deſcribed by the arch AE by revolving about the axis AB; and, if R be a mean proportional betwixt 2VK and the ſum of the right line AR added to the arch AE or *ae*, the area A ſhall be equal to the ſurface deſcribed by the arch *ae* about AB. But, if E be taken betwixt A and L; then, according as R is a mean proportional betwixt 2VK and the exceſs of the aroh AE above AR, or betwixt 2VK and the ſum of AE and AR, the area A is equal to the ſurface deſcribed by AE, or that which is deſcribed by *ae*. For, let G*t* produced meet AR in *r*, and let E*e* meet L*l* in *d*; and, ſince R*r* is to DG as AR is to AD, or VA to VK, and DG is to E*t* as *d*E is to VE, it follows, that R*r* is to E*t* as *d*E is to VK; and that VK is to DE as E*t* is to the difference of R*r* and E*t*, but that VK is to D*e* as E*t* is to the ſum of R*r* and E*t*. Therefore the fluxion of the ſurface deſcribed by AE is meaſured by a circle whoſe radius is a mean proportional betwixt 2VK and the difference of R*r* and E*t*, and the fluxion of the ſurface deſcribed by the arch *ae* is meaſured by a circle whoſe radius is a mean proportional betwixt 2VK and the ſum of R*r* and E*t*. But R*r* meaſures the fluxion of AR, and E*t* meaſures the fluxion of the arch AE or *ae*. Therefore, according as R is a mean proportional betwixt 2VK and the difference of AR and AE, or betwixt 2VK and the ſum of AR and AE, the area A is equal to the ſurface deſcribed by the arch AE, or that which is deſcribed by *ae*. When E is upon the leſſer arch ſubtended by the chord AB, the ſum of the ſurfaces deſcribed by AE and *ae* is equal to the area of a circle

whose radius is a mean proportional betwixt 2VK and 2AR, or betwixt the diameter and 2AD; but when E is betwixt A and L, the sum of the surfaces described by AE and *ae* is equal to a circle whose radius is a mean proportional betwixt 4VK and the arch AE. Let *b*B produced meet AR in M, and the surface described by the whole circumference of the circle, by revolving about the chord AB, is equal to a circle the radius of which is a mean proportional betwixt 4VK and the sum of the right line AM and arch AL*a*. The difference of the surfaces described by the arches A*ab*B and AEB is equal to a circle whose radius is a mean proportional betwixt 2VK and the whole

FIG. 77. circumference. When the axis AB does not meet the circle, let L*l* be the diameter parallel to AB, and let LA, *l*B, VK, *d*D be perpendicular to the axis in A, B, K and D. Let E be in the upper, and *e* in the lower semicircle; and, according as the square of R is equal to the sum or difference of the rectangles contained by 2VK and the arch LE, and by 2VL and AD, the area A is equal to the surface described by LE, or that which is described by L*e*. The surface described by the arch EL*e* is equal to a circle the radius of which is a mean proportional betwixt this arch and 2VK; and the surface described by the whole circumference is equal to a circle the radius of which is a mean proportional betwixt the circumference and 2VK. When the axis AB touches the circle, this radius is a mean proportional betwixt the circumference and diameter.

FIG. 78. n. 1, 2. 233. COR. III. In general, let F*f* be an arch of any curve line, let *ab* parallel to AB meet DE in *d*; and the area of a circle whose radius is a mean proportional betwixt 2D*d* and the arch F*f* shall be equal to the sum of the surface described by the arch F*f* revolving about the axis AB added to the surface described by the same arch revolving about *ab*, when the arch F*f* is betwixt AB and *ab*, but equal to the difference of those surfaces when AB and *ab* are on the same side of the arch F*f*; because, in the former case, the rectangle contained by D*d* and E*t* is equal to the sum of the rectangles DE*t*, *d*E*t*, and in the latter case it is equal to the difference of those rectangles: and therefore the sum of the fluxion of the surface described by FE about AB, added to the fluxion of the surface described by FE about


ig. 68
Fig. 69
Fig. 70.
Fig. 71.
Fig. 74
Fig. 68. n. 2.
Fig. 72.
Fig. 77.
Fig. 76. n. 2.
Fig. 75.

about *ab* in the former caſe, and the difference of thoſe fluxions in the latter caſe, is equal to the fluxion of an area that is always equal to a circle whoſe radius is a mean proportional betwixt 2D*d* and the arch FE: From which it follows, (by theor. 8.) that this circle is equal to the ſum of the ſurfaces deſcribed by FE about AB and *ab* in the firſt caſe, and to their difference in the ſecond caſe.

234. Cor. IV. It follows from the laſt corollary, that, if the axis *ab* meet the arch F*f* in *a*, ſo that the part F*a* be on the ſame ſide of *ab* with the axis AB, and *af* be on the oppoſite ſide of *ab*; then a circle whoſe radius is a mean proportional betwixt 2D*d* and the arch F*f* ſhall be equal to the ſurface deſcribed by F*f* about AB, added to the exceſs of the ſurface deſcribed by F*a* about *ab* above that which is deſcribed by *af* about the ſame axis *ab*. Therefore, if the axis *ab* cut the arch F*f* in ſuch a manner in the point *a*, that the ſurfaces deſcribed by F*a*, *af* (the parts of F*f* that are on oppoſite ſides of *ab*) about the axis *ab* be equal, then the ſurface deſcribed by F*f* about AB ſhall be equal to a circle whoſe radius is a mean proportional betwixt 2D*d* and the arch F*f*. And, converſely, if the ſurface deſcribed by F*f* about AB be equal to ſuch a circle, the ſurfaces deſcribed by the parts of F*f* that are on different ſides of *ab* muſt be equal to each other. Fig. 79.

235. Cor. V. Let AB meet the arch F*f* in A, ſo that the ſurfaces deſcribed by FA, A*f* (the parts of F*f* that are on different ſides of AB) revolving about AB may be equal to each other; let *ab* perpendicular to AB meet the arch F*f* in *a*, ſo that the ſurfaces deſcribed by F*a*, *af* (the parts of F*f* that are on different ſides of *ab*) about *ab* may be alſo equal to each other; and let *ab* meet AB in C. Then, if the ſame arch F*f* revolve about any axis CN that paſſes through this interſection C, the ſurfaces deſcribed by FN, N*f* (the parts of F*f* that are on different ſides of this axis CN) ſhall be alſo equal to each other. For, let CN meet the arch F*f* in N; let K be any point given upon CN, and KI be perpendicular on AB in I. Let E be any point of the arch F*f*, and let ED, EP, EM be perpendicular on KN, *ab*, AB in D, P and M reſpectively; and let EP (produced if neceſſary) meet KN in R. The right Fig. 80.

 line

line ER is the difference of EP and PR when the point E is taken upon the arch FA or *af*, in the case represented by fig. 80. and ER is the sum of EP and PR when E is taken upon the arch A*a*. But ED is always to ER as KI is to CK; and, since PR is to CP (or EM) as CI is to KI, it follows, that the rectangle contained by CK and DE is equal to the difference or sum of the rectangle contained by KI and PE and that which is contained by CI and ME; and, the fluxion of the arch AE being represented by E*t*, the solid contained by CK and the rectangle DE*t* is equal to the difference or sum of the solid contained by KI and the rectangle PE*t* and that which is contained by CI and the rectangle ME*t*, according as E is upon the arches F*a*, *af*, or upon A*a*. Hence the solid contained by CK and the surface described by the arch AE (or *a*E) revolving about the axis KN, is equal to the difference of the solid that is contained by KI and the surface described by AE (or *a*E) about *ab* and the solid contained by CI and the surface described by AE (or *a*E) about AB, when E is taken upon the arch FA or *af*, but to the sum of those solids when E is taken upon the arch A*a*. Therefore the solid contained by CK and the surface described by the arch FN about the axis KN is equal to the difference of two solids, the first of which is that contained by KI and the difference of the surfaces described by F*a* and *a*N about the axis *ab*, the second is the solid contained by CI and the difference of the surfaces described by FA and AN about the axis AB. And, in like manner, the solid contained by CK and the surface described by the arch N*f* about KN is equal to the excess of the solid contained by KI and the surface described by N*f* about *ab* above the solid contained by CI and the surface described by N*f* about AB. But it follows from the supposition, that the difference of the surfaces described by the arches F*a* and *a*N about *ab* is equal to the surface described by the arch N*f* about the same axis *ab*; and that the difference of the surfaces described by FA and AN about AB is equal to the surface described by N*f* about AB. Therefore the surfaces described by the arches FN, N*f* about the axis KN are equal to each other when the surfaces described by FA and A*f* about AB are equal, and the surfaces described by F*a* and *af* about *ab* are equal.

qual at the ſame time. This demonſtration is eaſily applied to the other caſes that are not repreſented in fig. 80.

236. Cor. VI. It follows from what has been demonſtrated, that, any arch *Ff* being given, there is a certain point C in the plane of this arch, through which any right line KN being drawn meeting the arch in N, the ſurfaces deſcribed by FN, N*f* (the parts of the arch that are on oppoſite ſides of KN) revolving about the axis KN are always equal to each other. This point is called the *center of Gravity* of the arch, (becauſe, if it be ſuppoſed to conſiſt of matter that is acted upon by an uniform gravity in parallel lines, the *momentum* of the part FN about any axis KN that paſſes through C is equal to the *momentum* of the part N*f* about the ſame axis, as we may have occaſion to ſhew afterwards;) and it follows from the two laſt corollaries, that the ſurface deſcribed by the arch F*f* revolving about *kn*, perpendicular to any line C*c* drawn through C, is equal to the area of a circle whoſe radius is a mean proportional betwixt 2C*c* and the arch F*f* when *kn* does not cut the arch F*f*; but the difference of the ſurfaces deſcribed by the parts of the arch F*f* that are on oppoſite ſides of *kn* is equal to that circle when *kn* cuts the arch F*f*. The area of ſuch a circle is equal to the rectangle contained by the arch F*f* and the circumference of the circle deſcribed by the point C when the figure is ſuppoſed to revolve about *kn*; and therefore this corollary agrees with the celebrated theorem commonly aſcribed to Guldinus.

237. Cor. VIII. Hence the diſtance of the point C from any right line *kn* is determined, when the length of the arch F*f* and the ſurface deſcribed by this arch revolving about *kn* are given. For a third proportional to the length of the arch F*f* and the radius of a circle that is equal to that ſurface ſhall be double of the diſtance of C from *kn*. Thus, becauſe the ſurface deſcribed by the arch AEB, revolving about the diameter L*l* parallel to the chord AB, is equal to the area of a circle whoſe radius is a mean proportional betwixt the diameter L*l* and chord AB, it follows, that the diſtance of the center of gravity of the arch AEB from the center of the circle is to the radius as the chord AB is to the arch AEB. Fig. 76.

CHAP.

CHAP. IX.

Of the greatest and least Ordinates, of the points of contrary Flexion and Reflexion of various kinds, and of other affections of Curves that are defined by a common or by a fluxional Equation.

238. THere are hardly any ſpeculations in Geometry more uſeful or more entertaining * than thoſe which relate to the *Maxima* and *Minima*. Several propoſitions of this nature are to be found in the writings of the ancient Geometricians; but they do not ſeem to have had a general method for reſolving problems of this kind. Amongſt the various improvements that began to appear in the higher parts of Geometry about a hundred years ago, Mr. de FERMAT propoſed a method for finding the *maxima* and *minima*. How the methods that were then invented for the menſuration of figures and drawing tangents to curves are comprehended and improved by the method of Fluxions, may be underſtood from what has been already demonſtrated. A general way of reſolving queſtions concerning the *maxima* and *minima* is alſo derived from it, that is ſo eaſy and expeditious in the moſt common caſes, and is ſo ſucceſsful when the queſtion is of a higher degree, where the difficulty is greater, and other methods fail us, that this is juſtly eſteemed one of the moſt admirable applications of Fluxions.

239. When the nature of a variable quantity is ſuch, that it either increaſes continually without end, or decreaſes till it vaniſhes, its greateſt or leaſt magnitude is not aſſignable; and there is no place for enquiries of this nature. But, when there is a certain limit which the increaſe or decreaſe of the variable quantity cannot paſs, and the term is aſſignable when it arrives at this limit; or, more generally, when for ſome time the variable quantity firſt increaſes till a certain aſſignable term, and then

* Apoll. Perg. Conic. lib. 5. præf.

decreaſes,

decreases, or first decreases till such a term and then increases: its magnitude at that term is considered as a *maximum*, or *minimum*, without regard to its variations in other parts of the time.

240. In the problems of this kind of the first degree, the variable quantity is represented by an ordinate of a curve the nature of which is supposed to be defined by what is given concerning the variable quantity. A curve line either returns into itself, or may be continued without end; and therefore there are always two branches of the curve that proceed from any point that is assignable in it. The ordinate from a point of the curve is a *maximum*, or *minimum*, when it is greater or less than the ordinates which may be drawn from the parts of either branch of the curve adjoining to that point. When the curve is continued immediately from that point on both sides of the ordinate, we shall call the ordinate a *maximum* or *minimum* of the *first kind*, but of the *second kind* when the curve is reflected from that ordinate and both the branches of the curve are on the same side of it. Sometimes three or more branches of a curve proceed from the same point; but the ordinate in that case is only to be compared with the ordinates from these two branches that are to be considered, according to the nature of the curve, as the immediate continuation of each other. FIG. 81. & 82. FIG. 83.

241. The writers on this subject do not always agree as to the extent of the problems they comprehend under this class. Some * have proposed to comprehend under the *maxima* and *minima* the greatest or least ordinates that can be drawn from the part of the curve that is convex towards the base, or from that which is concave towards the base, though the ordinates from the adjoining parts of the succeeding branch of the curve may be greater or less than that ordinate. Others, who exclude that case, seem also to exclude the greatest or least ordinate when it passes through a point where the curvature is not continued, that is, a point of *Reflexion*, or *Cuspis*. Others comprehend the ordinate from a cuspis when the two branches of the curve that proceed from it are on different sides of the ordinate, but exclude such an ordinate when the two branches of

* Descartes's letters, tom. 3. let. 60.

the

the curve that proceed from the cuspis are on the same side of the ordinate, because those branches (or at least the adjoining parts of each) are over the same base. But it seems to be more consistent, to include all ordinates that are greater, or less, than those from the adjoining parts of either branch of the curve; FIG. 83. the rather, that in the latter case one of the branches of the curve, after being reflected from the ordinate, often returns to it again, and after cutting it, proceeds on the other side, so that one or more ordinates may correspond to any assignable base. However, since it has been more usual to exclude this case, we have distinguished the greatest and least ordinates into two kinds in the preceeding article, to prevent mistakes.

FIG. 81. 242. When an arch of a curve has its concavity turned one way, and there is a point in this arch where the tangent becomes parallel to the base, the greatest or least ordinate passes through that point. It is greater or less than those from the adjoining parts of the arch on either side, according as the arch is concave or convex towards the base. Supposing the base to increase, the ordinate in the former case first increases and then decreases, and in the latter case the ordinate first decreases and then increases. In both cases, the motion with which the base flows (or its fluxion) being given, the motion with which the ordinate flows (or its fluxion) first decreases and then increases, by the 7th lemma: And this motion, or the fluxion of the ordinate, vanishes when the tangent becomes parallel to the base. For, EI (fig. 47. & 50.) being supposed to measure the fluxion of the base, IT which measures the fluxion of the ordinate (by prop. 14.) vanishes when the tangent ET becomes parallel to the base and coincides with EI. Therefore the greatest and least ordinates are discovered in such cases, (which are those that most commonly occur in the resolution of problems,) by enquiring when the fluxion of the curve becomes equal to the fluxion of the base, or when the fluxion of the ordinate vanishes the fluxion of the base being given. In this case however the curve must be continued on both sides of the ordinate: for, if the curve be reflected from the ordinate DE, then the point E is a cuspis, and the ordinate DE is neither a *maximum* nor *minimum* when the arches CE, EH have their convexity towards

wards each other; and it is a *maximum* or *minimum* of the ſe- FIG. 84.
cond kind when thoſe branches are on the ſame ſide of the tan- FIG. 85.
gent at E, the convexity of one branch being towards the concavity of the other.

243. The arch being ſuppoſed to have its concavity turned one way, and the tangent at E being ſuppoſed parallel to the FIG. 81.
baſe, if the arch meet the baſe we may conclude that DE is a n. 1.
maximum; but if the arch be of that kind which may be con- FIG. 81.
tinued above the baſe without end, DE is a *minimum*. The n. 2.
greateſt and leaſt ordinates are diſtinguiſhed from each other more generally, by comparing them with the ordinates from the adjoining parts of the curve; or by ſuppoſing the baſe AP to increaſe, and obſerving whether the fluxion of the ordinate PM is poſitive before the point P comes to D and becomes negative after P paſſes D, or is firſt negative and then becomes poſitive, the ordinate being itſelf conſidered as poſitive: It is a *maximum* in the former, and a *minimum* in the latter caſe.

244. In the application of this rule for finding the greateſt and leaſt ordinates, it muſt be obſerved, that the ratio of the fluxion of the ordinate to the fluxion of the baſe may be ſometimes repreſented by the ratio of two quantities which at certain terms may both vaniſh together: but it does not follow, that in ſuch caſes the tangent becomes parallel to the baſe, or that the ordinate is a *maximum* or *minimum*; for the ratio of thoſe fluxions may be repreſented in ſuch caſes by that of other finite and aſſignable quantities. When two or more branches of the curve are over the ſame part of the baſe, the ordinate muſt ariſe to two or more dimenſions in the equation of the curve. The fluxion of the baſe being given, the fluxion of the ordinate is at leaſt twofold; and when the ordinate paſſes through a point where two or more branches of the curve interſect each other, we are to expect, that though the baſe and ordinate be both given, the fluxion of the ordinate will ariſe to two or more dimenſions in the general equation by which it is to be determined. But if the general equation of the curve can be reſolved into theſe particular equations which belong to the different branches of the curve from which the general equation is compounded, then the fluxion of each ordinate may be determined

 from

from the fluxion of the equation of each branch. But we may have occasion to illustrate this further afterwards.

FIG. 82. 245. When the fluxion of the curve coincides with the fluxion of the ordinate, the tangent becomes perpendicular to the base, and coincides with the ordinate. In this case, if EC, EH the two branches of the curve form a cuspis at E, being on different sides of the ordinate DE, then is DE a *maximum* or *minimum* according as those branches have their convexity or concavity towards the base. The *maxima* and *minima* of this sort are discovered, by enquiring when the fluxion of the curve becomes equal to the fluxion of the ordinate, or (which is the same thing) when the fluxion of the base vanishes the fluxion of the ordinate being given. In this case also it is necessary, that the curve be continued immediately from E on both sides of the ordinate DE; for if the two branches EC, EH be on the same side of DE, the ordinate is either no *maximum* nor *minimum*, or is one of the second kind.

246. These are the two rules that are commonly given for determining the *maxima* and *minima* of the first kind. But there is still another limitation besides those already mentioned, without which these rules may lead us into error. For we are not always to conclude that the ordinate DE is a *maximum* or *minimum*, either when the fluxion of the ordinate vanishes the fluxion of the base being given, or when the fluxion of the base vanishes the fluxion of the ordinate being given, though the curve be continued immediately on both sides of the ordinate: In the former case the tangent at E is parallel, in the latter perpendicular to the base; but E may be a point of contrary flexion, so that the ordinates on one side of DE may be greater than DE, and those on the other side less than it; and there may be no *maximum* nor *minimum* perhaps from the whole curve. In any curve that has a point of contrary flexion, the ordinate discovered by those rules is not a *maximum* or *minimum* if the base be parallel or perpendicular to the tangent at that point. The rules which are given for finding the points of contrary flexure are liable to exceptions of the same nature for a similar reason. In order to set these rules and the necessary exceptions in a clear light, it will be of use to premise the following proposition.

FIG. 91.

PROP.

PROP. XX.

247. *Let the ordinate* DE *of the curve* AEH *meet the curve* Feh *in* e, *and the rectangle* EG *contained by* DE *and a given right line* DG *be always equal to the area* ADeF; *then, the rectangle* DeLG *being completed,* ET *the tangent of the curve* AEH *at* E *shall be parallel to the diagonal* DL. *And, conversely,* EK *being equal and parallel to* DG, *if* KT *parallel to* DE *meet the tangent* ET *in* E, *and,* De *being taken always equal to* KT, *if the curve* HEC *meet the base in* A, *the area* ADeF *shall be equal to the rectangle* EG. Fig 86.

The fluxion of the base AD being represented by the given right line DG or EK, the fluxion of the ordinate DE is represented by KT, by prop. 14. and the fluxion of the area AD*e*F by the rectangle *e*G, by prop. 3. The rectangle EG is always equal to the area AD*e*F by the supposition; and the fluxion of EG is equal to the fluxion of the area AD*e*F, by art. 18. that is, the rectangle contained by KT and DG is equal to *e*G, and KT is equal to D*e* or GL. Therefore the tangent ET is parallel to DL.

248. And, conversely, the right line DG or EK being given, if upon the ordinate DE a right line D*e* be taken always equal to KT, the rectangle *e*G shall be always equal to the rectangle contained by KT and DG; that is, the fluxion of the area AD*e*F shall be always equal to the fluxion of the rectangle EG; and, by theor. 4. the fluents generated in the same time being equal, it follows, that since the curve CEH is supposed to pass through A, and the point E sets out from A when *e* sets out from F and the right line D*e* from AF, the area AD*e*F must be equal to the rectangle EG. The analogy there is betwixt the inverse method of tangents and the quadrature of curvilineal figures appears from this proposition.

249. It may be of use for illustrating this doctrine, to demon-

strate in the following manner, (which is independent of the method of fluxions,) that if the rectangle EG be always equal to the area AD*e*F, then ET parallel to DL shall be the tangent of the arch EH at E. For, let any ordinate PM meet the arches EH, *eh*, the right lines EK, *e*L parallel to the base, and ET parallel to DL, in the points M, N, V, S and R respectively. Then, since the rectangle contained by VR and DG is to the rectangle contained by KT and DG as VR is to KT, or DP to DG, or as the rectangle *e*P is to *e*G; and the rectangle contained by KT and DG is equal to *e*G; it follows, that the rectangle contained by VR and DG is equal to the rectangle *e*P. But by the supposition the rectangle contained by PM and DG is equal to the area APNF; and the rectangle contained by DE and DG is equal to the area AD*e*F: therefore the rectangle contained by VM and DG is equal to the area DPN*e*; and VM is to VR as the area DPN*e* is to the rectangle *e*P. But when the ordinates from the arch *eh* increase while the base increases, the area DPN*e* always exceeds the rectangle *e*P. Therefore in this case VM is always greater than VR, and the arch EH is all above the right line ET, the point E only excepted. Nor can any right line be drawn through E within the angle of contact HET. For, since KH is to KT (or GL) as the area DG*he* is to the rectangle *e*G, and therefore in a less ratio than G*h* is to GL, it follows, that KH is less than G*h*. Therefore, Q being any point betwixt T and H, if DO be taken equal to KQ, it shall be less than G*h*, and a right line through O parallel to the base shall meet the arch *eh* in some point betwixt *e* and *h*. Let this point be N, and let the ordinate PN meet the arch EH and the right lines EQ, EK, *e*L in the points M, Z, K and S. Then, since the rectangle contained by KQ and DG is to the rectangle contained by VZ and DG as KQ is to VZ, or DG to DP, or as the rectangle OG is to OP, it follows, that the rectangle contained by VZ and DG is equal to OP, and that VZ is to VM as the rectangle OP is to the area DPN*e*. Therefore VZ is greater than VM, and the right line EQ does not pass through the angle of contact HET, but cuts the arch EH in some point betwixt M and H. From which it follows, by art. 181. that ET is the tangent of the arch EH at E.

250. In

250. In the same manner, if any ordinate *pn* meet the arches *ce*, CE and the right lines E*t*, E*k*, *el* (which are the right lines TE, KE, L*e* continued beyond E and *e*) in *n*, *m*, *r*, *v* and *s*, *vm* shall be to *vr* as the area D*pne* is to the rectangle *ep*; and therefore, when the ordinates from *ce* increase while the base increases, the arch EC is all above E*t*, the point E excepted. In the same manner, *k*C is greater than B*c*, (the rectangle contained by *k*C and DG being equal to the area DB*ce* which exceeds the rectangle contained by B*c* and DG;) and *kt* being equal to KT or D*e*, if *q* be any point betwixt C and *t*, and D*o* be taken equal to *kq*, a parallel to the base through *o* shall meet the arch *ce* in some point betwixt *c* and *e*. Let that point be *n*, and let the ordinate *pn* meet E*q* in *z*; then, since *kq* (or D*o*) is to *vz* as DB is to D*p*, or as the rectangle *o*B is to *op*, it follows, that the rectangle contained by *vz* and DG is equal to the rectangle *op*, and that *vz* is to *vm* as *op* is to the area D*pne*. Therefore *vz* is less than *vm*; and, since *k*C is less than *k*Q, the right line E*q* does not pass through the angle of contact CE*t*, but intersects the arch EH in some point betwixt *m* and C. Therefore E*t*, which is the right line ET continued, is the tangent of the arch EC which is the continuation of the arch HE. Because the tangent *t*T is betwixt the curve CEH and the base in this case, it appears, that when the ordinates from the arch *ceb* increase, (the base AP being supposed to increase,) the arch CEH is convex towards the base.

251. When the ordinates from the arch *ceb* decrease while the base increases, the rectangle *e*P exceeds the area DPN*e*, and the rectangle *ep* is less than the area D*pne*. Therefore VR is greater than VM, and *vr* less than *vm*; so that the whole arch HEC is below the right line TE*t*, the point E excepted: And it is shewn in the same manner as in the preceeding case, that no right line can be drawn through the angles of contact HET, CE*t*. Therefore, in this case, the right line T*t* is the tangent at E, and the arch CEH is concave towards the base. In both cases the curve CEH passes through A, because when AP vanishes, the rectangle contained by PM and DG (which is supposed to be always equal to the area APNF) vanishes; and if A*b* be taken towards G equal to DG, and the rectangle A*bf*F FIG. 87.

be-

be completed, the diagonal A*f* shall be the tangent at A.

FIG. 86. 252. COR. I. Let Z*ez* the tangent of the arch *ceb* at *e* meet G*b* and B*c* in Z and *z*, and let the ordinates from B*c* to G*b* increase continually; then, if the arch *ceb* be convex towards the base, TH which subtends the angle of contact HET shall be less than one half of L*b*, but greater than one half of LZ. For the rectangle contained by TH and DG is equal to the area *e*L*b* which is less than the triangle *e*L*b*, or one half of the rectangle contained by DG and L*b*, but is greater than the triangle *e*LZ, or one half of the rectangle contained by DG and LZ. Therefore TH is less than one half of L*b*, but greater than one half of LZ. In the same manner it appears, that *t*C is less than one half of *lz*, but greater than one half of *lc*.

FIG. 87. 253. COR. II. When the arch *ceb* is concave towards the base, TH is less than one half of LZ, but greater than one half of L*b*. For, in this case, the area *e*L*b* is less than the triangle *e*LZ, but greater than the triangle *e*L*b*. In like manner *t*C is in this case less than one half of *lc*, but greater than one half of *lz*.

254. COR. III. When *ceb* is a right line and coincides with *ze*Z, CEH is an arch of a parabola that has its axis perpendicular to the base AG. In this case TH and *t*C are each equal to one half of LZ which measures the fluxion of D*e* (by prop. 14) or the second fluxion of DE, the fluxion of the base being represented by DG. While the base AD acquires the augment DG, the ordinate DE acquires the augment KH equal to the sum of KT and TH; and in this case the first fluxion of the ordinate is represented by KT, and its second fluxion by 2TH, or the sum of TH and *t*C. But when *ceb* is convex towards the base, and, the base being supposed to flow uniformly, its fluxion is represented by DG, the right line TH is greater than one half of LZ which measures the second fluxion of the ordinate DE, but less than one half of L*b* which measures the increase of the fluxion of the ordinate that is generated in the same time in which the base acquires the augment DG. When the arch *ceb* is concave towards the base, TH (which subtends the angle of contact HET) is less than one half of the right line that measures the second fluxion of DE, but greater than one half of the right line that measures the increase of the fluxion

xion of DE. In the first case, when *ceb* is a right line, the motion with which DE flows is uniformly accelerated. When *ceb* is convex towards the base, the acceleration of that motion increases; and when *ceb* is concave towards the base, its acceleration decreases continually.

255. Cor. IV. When the curve A*cb* is a parabola that has its axis perpendicular to the base AG, KH the increment of DE may be distinguished into three parts, KT, TQ and QH, so that the rectangle contained by those parts and DG may be respectively equal to the rectangle *e*G, the triangle *e*LZ, and the area *e*Z*b*. The part KT is equal to D*e*, and measures the first fluxion of DE; the part TQ is equal to one half of LZ, which measures the second fluxion of DE, (by cor. 3.) and the part QH is equal to one third part of Z*b*, (that measures one half of the fluxion of LZ,) and therefore measures one sixth part of the third fluxion of DE. For it follows from what was shewn in the Introduction (pag. 27.) after Archimedes, that the area *e*Z*b* is one third part of the rectangle contained by Z*b* and DG: And it may be easily deduced from the 8th proposition; for let PM meet the arch *eb* in N, and its tangent *e*Z in *u*; and *u*N shall be to Z*b* as the square of DP is to the square of DG. Therefore, the point D and the right lines DG, Z*b* being given, but supposing DP to flow, and DG, DP, X and Y to be in continued proportion, it will follow from the eighth proposition, that one third part of the fluxion of Y shall be to the fluxion of DP as Y is to DP, or as the square of DP is to the square of DG, and therefore as *u*N is to Z*b*; so that the rectangle contained by *u*N and the right line which measures the fluxion of DP is equal to one third part of the rectangle contained by Z*b* and the right line which measures the fluxion of Y. Therefore the area *e*UN is equal to one third part of the rectangle contained by Z*b* and Y, or (because Z*b* is to *u*N as DP is to Y) of the rectangle contained by *u*N and DP; and the area *e*Z*b* is equal to one third part of the rectangle contained by Z*b* and DG. In the same manner it is shewn, that the area ERM is equal to one fourth part of the rectangle contained by RM and DP; and the continuation of these theorems is obvious from the same eighth proposition. From which it follows, Fig. 88.

lows, that when the fluxions of all orders of the ordinate DE increase, we approximate continually to the value of KH, the increment of the ordinate that is generated in the same time the base acquires the augment DG, by adding continually together the right line that measures the first fluxion of DE, while DG measures the fluxion of the base, $\frac{1}{2}$ of that which measures the second fluxion of the ordinate, $\frac{1}{6}$ of that which measures its third fluxion, $\frac{1}{24}$ of that which measures its fourth fluxion, and so on, the denominators of those fractions being the products of the numbers 1, 2, 3, 4, 5, &c. in their natural order. But when any fluxion decreases, the succeeding fluxion is to be considered as negative, and the fraction which involves it is to be subducted. These corollaries illustrate what was shewn of second and third fluxions near the end of the first and fourth chapters.

FIG. 89. 256. Let any point E be given in the curve CEH; let KE*k* be a right line parallel to the base AD, PM an ordinate meeting EK in V and the curve *ceb* in N, and the rectangle contained by VM and the given right line DG be always equal to the area DPN*e*. Suppose the points P and M to move from D and E; and, when AP increases and PN is above the base AP, let VM be taken upon PV produced beyond V: then, when AP decreases, if PN be still above the base, or when AP increases if PN be below the base, VM is to be taken upon VP from V towards P; but if AP decrease and PN be below the base, VM is to be taken upon PV produced beyond V.

FIG. 90. 257. When the point *e* in the curve F*cb* falls on the base and coincides with D, DL coincides with DG, the tangent *t*ET coincides with *k*EK and becomes parallel to the base. In this case, if the curve F*cb*, after meeting the base AD, be continued on the other side of AD and on the other side of the perpendicular DE, then is DE a *maximum*; for, by the last article, P being taken on either side of D, VM is to be taken from V towards P. When PM meets the curve F*cb* below the base in N and the curve CEH above it in M, the rectangle contained by VM and DG being equal to the area DPN, it follows, that the rectangle contained by PM and DG is equal to the excess of the area ADF above the area DPN, and that PM

PM vanishes when those areas become equal. If the curve FDN return towards the base, and after cutting it again in d be continued on the opposite side of the base and of the perpendicular de, then de the ordinate of the curve AEM at d is a *maximum* or *minimum* according as it meets the curve above or below the base. In the same manner, if the curve NF continued beyond F meet the base in *a*, and proceed from *a* below the base on the other side of the perpendicular at *a*, the curve MEA shall be continued from A below the base, and its ordinate at *a* shall be a *maximum*.

258. By the 256th article, the form of the arch *ceb* being given, we discover the form of the arch CEH. Suppose now the form of the arch CEH to be known, and that of *ceb* to be required. Let E be any point in the arch CEH, and suppose that D*e* is taken above the base when KT is upon GK produced beyond K. Then, if the tangent ET meet GK in T betwixt G and K, or B*k* produced beyond *k* in *t*, the ordinate D*e* is to be taken upon ED produced beyond D below the base; but when ET meets B*k* in *t* betwixt B and *k*, D*e* is to be taken above the base. FIG. 89.

259. From this the converse of the 257th article is manifest: That when the ordinate DE is a *maximum* or *minimum* of the first kind, and the tangent at E is parallel to the base, the arches *ec*, *eb* of the curve *ceb* must be continued from *e* (which coincides with D in this case) on different sides of the base AD and of the perpendicular at D. For, while the point M describes the arches CE, EH, the ordinate PN of the curve *ceb* must be taken on different sides of the base. FIG. 90.

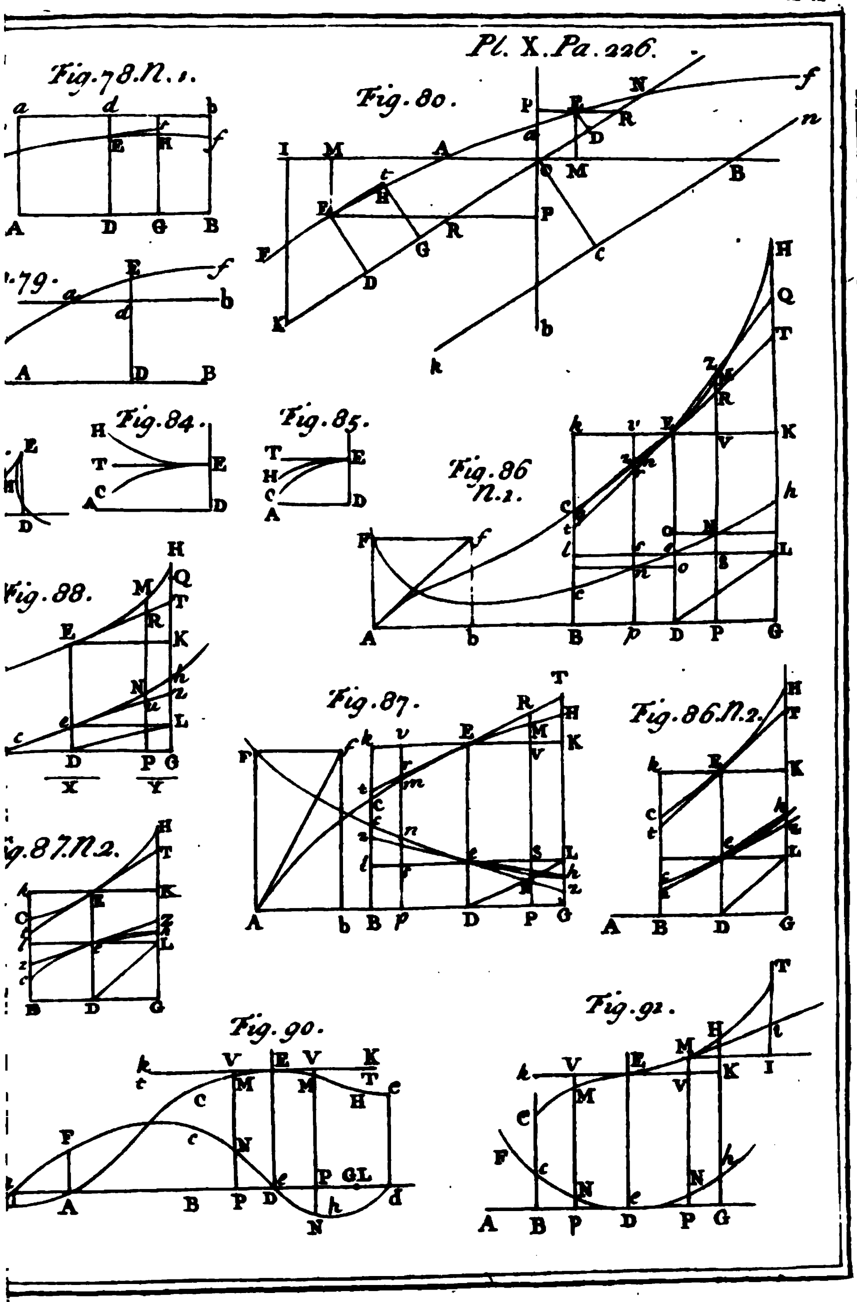
260. But when the curve FND, after meeting the base in D, is continued on the opposite side of DE, but on the same side of the base as before, then the ordinate DE is not a *maximum* or *minimum*, though PN which measures the fluxion of the ordinate vanishes when P comes to D. For, while N describes the archs *c*D and D*b*, the right line VM is to be taken on different sides of *k*K, which is parallel to the base. Therefore DE is not a *maximum* or *minimum*, but E is a point of contrary flexure, whether the arch *c*D*b* touch the base in D, or form a cuspis there. It is manifest, conversely, from art. 258. that, when E FIG. 91.

is a point of contrary flexure and the tangent at E is parallel to the base, the arch *cb*, after meeting the base in D, is continued on the same side of the base; because the ordinates from BC to GH increase or decrease continually: and, MI being equal and parallel to DG, if IT parallel to DE meet the tangent at M in t, the right line It is on the same side of MI while M describes the arches CE and EH.

PROP. XXI.

261. *The fluxion of the base being given, and the curve being continued on both sides of the ordinate, when the first fluxion of the ordinate and its fluxions of any number of subsequent successive orders vanish, the ordinate is a* maximum *or* minimum, *or passes through a point of contrary flexion, according as that number is even or odd.*

Fig. 90. & 91. The curve being continued on both sides of the ordinate, and the fluxion of the base being given, let the first fluxion of the ordinate DE vanish; then, if the number of its subsequent successive fluxions that vanish be 0, 2, 4, or any even number, the ordinate DE shall be a *maximum* or *minimum*; but if that number be 1, 3, 5, or any odd number, E shall be a point of contrary flexure First, let the curve *cb* cut the base in D in any assignable angle, and be continued from D on opposite sides of the base and ordinate. In this case DE is a *maximum* or *minimum* by art. 257. and the first fluxion of DE vanishes, but the second fluxion does not vanish. In the next place, let the arch CH be substituted in place of *cb*, and a new curve be derived from CH in the same manner that CH was derived from *cb*: Then DE shall meet this third curve in a point of contrary flexure, by art. 260. and the first and second fluxions of the ordinate of this curve vanish. If this third curve be substituted for *cb*, and a fourth be derived from it in the same manner, the ordinate of this fourth curve at D shall be a *maximum* or *minimum*, and its first, second and third fluxions, vanish.


Pl. X. Pa. 226.
Fig. 78. N. 1.
Fig. 80.
.79.
Fig. 84.
Fig. 85.
Fig. 86 N. 1.
Fig. 88.
Fig. 87.
Fig. 86. N. 2.
.87. N. 2.
Fig. 90.
Fig. 91.

nish. This series of curves being continued, it is manifest that DE shall be a *maximum* or *minimum*, and pass through a point of contrary flexure, alternately; the number of fluxions of successive orders (not including the first fluxion) that vanish being alternately an even and an odd number.

262. When *De* is an asymptote of the curve F*c*N, there are always two branches of the curve, as *c*N, *b*N, which approach to the asymptote when they are produced. If the area bounded by the base AD, the ordinate AF, the asymptote *De* and the curve F*c*N, is always less than a certain finite space to which that area continually approaches while the curve and its asymptote are produced, so that their difference may become less than any assignable space; and if the two branches *c*N, *b*N proceed alongst the asymptote on different sides of it and in different directions, then the ordinate DE is a *maximum* or *minimum* such as was described in art. 245. When those branches approach to the asymptote on different sides of it, but proceed alongst it with the same direction, E is a point of contrary flexure, and DE is not a *maximum* or *minimum*. In both cases, the tangent at E is perpendicular to the base, and the fluxion of the base vanishes, when compared with the fluxion of the ordinate; but it is only in the former case, when the fluxion of the ordinate is positive on one side of DE and negative on the other side of it, that DE is a *maximum* or *minimum*. We have often observed, that the fluxion of a quantity is considered as positive when the quantity increases, but as negative when it decreases: And these fluxions are represented by right lines that stand on opposite sides of the base, the contrary positions of which answer to the opposite affections of the fluxions. It is common to the *maxima* and *minima* described in the 242d and 245th articles, that, the fluxion of the base AP being positive, the fluxion of the ordinate PM is positive on one side of DE and negative on the other: But in the former it vanishes when PM coincides with DE; and in the latter it is said to become infinitely great in that case, as the ordinate of a curve is said to become infinitely great when it coincides with an asymptote. And it is said in general, that a quantity that is positive becomes negative, either by decreasing and passing by nothing, or by in-

FIG. 92.

FIG. 93.

FIG. 94. creasing and passing by infinity: As, if we suppose a right line FQ revolving about a given point F always to intersect the right line O*o* in P, and FA to be perpendicular to O*o* in A, the right line AP, being first considered as positive, becomes negative by decreasing and passing by nothing, when FQ being supposed to move from FO towards F*o*, the intersection P passes through A to the other side of A; but AP is said to become negative by increasing and passing by infinity, when FQ moves in a contrary direction, and, after becoming parallel to O*o*, meets it again on the other side of A. It is usual to explain analogies betwixt figures in the common doctrine of curve lines in this manner, and it is often of use in transferring readily the properties of one curve to another that is under the same *genus*. Thus authors explain how the ellipse is transformed into a parabola, and thence into an hyperbola. But, lest conceptions or expressions of this kind should be excepted against, we have endeavoured to avoid them.

PROP. XXII.

263. *The ordinate meets the curve in a point of contrary flexure when its fluxion is a* maximum *or* minimum, *the fluxion of the base being given and the curve being continued on both sides of the ordinate.*

FIG. 95. Resuming the construction of the 249th and 251st articles, it follows from what was demonstrated there, that, when the ordinates of the arch *ce* increase from B*c* to D*e*, the arch CE is convex towards the base, and that when the ordinates from D*e* to G*b* decrease, the arch EH is concave towards the base; that is, when D*e* is a *maximum*, and the arches *ce*, *eb* are on different sides of D*e*, the point E is a point of contrary flexure. But D*e* represents the fluxion of the ordinate DE, the fluxion of the base being represented by the given right line DG. Therefore, when the fluxion of the ordinate is a *maximum*, and the curve is continued from the ordinate on both sides, it meets the curve

FIG. 96. in a point of contrary flexure. In like manner, when the ordinates from the arch *ce* decrease and those from *eb* increase, (that

(that is, when D*e* is a *minimum*,) the arch CE is concave and the arch EH is convex towards the base, by what was shewn in the 249th and 251st articles. Therefore, when the fluxion of the ordinate is a *minimum*, the fluxion of the base being given and the curve being continued from the ordinate on both sides, the ordinate meets the curve in a point of contrary flexure.

264. The proposition appears also from the converse of the 7th lemma, art. 184. For, if we suppose the fluxion of the ordinate DE to be a *maximum*, the fluxion of the ordinate PM must increase while M describes CE, and decrease while M describes EH. Therefore, by the converse of the 7th lemma, if PM increases while M describes CEH, the arch CE must be convex and EH concave towards the base; so that E must be a point of contrary flexure. If PM decrease while M describes CEH, the arch CE must be concave and the arch EH convex towards the base, and E a point of contrary flexure. In the same manner, when the fluxion of DE is a *minimum*, it appears that E is a point of the same kind. We do not comprehend under the *maxima* or *minima* quantities that vanish, or such as are supposed to exceed all assignable magnitude. Fig 95. Fig. 96.

265. Cor. I. As there are various kinds of *maxima* and *minima*, so there are various kinds of points of contrary flexure. As in the most common cases, the ordinate is a *maximum* or *minimum* when its fluxion vanishes, the fluxion of the base being given; so, when the second fluxion of the ordinate vanishes, the ordinate most commonly passes through a point of contrary flexure. But this is not universally true though the curve be continued on both sides of that ordinate. For, when the tangent of the curve *ceb* at *e* becomes parallel to the base, the second fluxion of DE, or the first fluxion of D*e*, vanishes, the fluxion of the base being given; but in this case *e* may be a point of contrary flexure, (art. 260.) and, D*e* (which measures the fluxion of DE) not being a *maximum* or *minimum*, the point E is not a point of contrary flexure, but the whole arch CEH has its concavity turned the same way. Fig. 97.

266. Cor. II. In general, whether the first fluxion of the ordinate vanish or not, the point E is a point of contrary flexure when the number of the subsequent successive orders of fluxions Fig. 95. 96. & 97.

xions of DE that vanish is an odd number, if the curve be continued from the ordinate on both sides; but E is not a point of contrary flexure, and the whole arch CEH has its concavity turned the same way when that number is even; for in the former case D*e* is a *maximum* or *minimum*, and in the latter case *e* is a point of contrary flexure, by art. 261.

267. When D*e* or the fluxion of the ordinate DE is such a *maximum* or *minimum* as was described in the 245th and 262d articles, E is a point of contrary flexure, and, the tangent at *e* being perpendicular to the base, the right line that measures the fluxion of the base vanishes in this case when that which measures the fluxion of D*e*, or the second fluxion of DE, is given. But it does not follow, conversely, that when this happens, E is always a point of contrary flexure, although the curve be continued on both sides of the ordinate DE. For, if *e* itself be
FIG. 98. a point of contrary flexure in the curve FN, then, though the tangent at *e* be perpendicular to the base, D*e* is not a *maximum* or *minimum*, and E in the curve CEH is not a point of contrary flexure; but the whole arch CH has its convexity or concavity towards the base, according as the ordinates of the arch *cb* continually increase or decrease from B*c* to G*b*.

268. Hitherto we have supposed the two arches CE and EH to be on different sides of DE. When these arches are on the same side of DE and have a tangent at E different from DE, then
FIG. 99. E is a point of reflexion, or cuspis. The celebrated author of
& 101. the *Analyse des infiniment petits* distinguishes those points into two kinds; the point E is a cuspis of the first kind when the arches EC, EH have their convexity towards each other, but of the second kind when the convexity of the one is towards the concavity of the other.

269. Let the arch *cb* meet the base in D, the points D and *e* being supposed to coincide; and let the arches *ce*, *eb* be on different sides of the base, but on the same side of DE. Then, whether the arches *ce*, *eb* be perpendicular to the base and form one continued arch *cb*, or touch the base and form a cuspis of
FIG. 99. the first kind at *e*, the point E is a cuspis of the first kind. For,
n. 1. & when PN is above the base, VM is to be taken upon VP from
2. V towards P, and when PN is below the base, VM is to be taken

taken upon PV produced beyond V, by art. 256. And, converſely, when E is a cuſpis of the firſt kind, if the tangent at E be parallel to the baſe, when the baſe AP increaſes the ordinate from one of the arches CE, HE increaſes and the ordinate from the other decreaſes. The fluxion of one of the ordinates is poſitive and the fluxion of the other is negative; ſo that the right lines which repreſent theſe fluxions muſt be taken on oppoſite ſides of the baſe; and *ce*, *eh* muſt either form a continued arch *ch*, or a cuſpis of the firſt kind at *e*. In both caſes the firſt fluxion of DE vaniſhes; and in the former, the fluxion of D*e*, or the ſecond fluxion of DE, being given, the fluxion of the baſe vaniſhes: In the latter caſe, the ſecond fluxion of DE vaniſhes as well as the firſt fluxion, and, the third fluxion of DE being compared with the fluxion of the baſe, the one vaniſhes when the other is given. In general, when E is a cuſpis of the firſt kind and the tangent at E is parallel to the baſe, the fluxions of DE of any number of ſucceſſive orders may vaniſh the fluxion of the baſe being given; but the fluxion of the next order to thoſe that vaniſh cannot be to the fluxion of the baſe in any aſſignable ratio, and is ſaid (according to the uſual language on this ſubject) to become infinitely great, in the ſame ſenſe as the ordinate of a curve is commonly ſaid to become infinite when it is ſuppoſed to coincide with an aſymptote.

PROP. XXIII.

270. *The arches ce, eh being on the ſame ſide of* D*e*, *the point* E *is a cuſpis of the firſt kind when ceh is a continued arch, but of the ſecond kind when e is a cuſpis of the firſt kind and the tangent at e is oblique to* D*e*, *or when e is itſelf a cuſpis of the ſecond kind.* FIG. 100.

Caſe 1. When *ceh* is a continued arch and touches the ordinate D*e*, VM is to be taken upon VP from V towards P both when N deſcribes the part *ce* and the part *eh*, by art. 256. The right line E*t* parallel to DL is the tangent of both the branches.

CE and EH, by prop. 20. Therefore the point E is a cuſpis. The areas D*ec*B, D*el*B, D*eb*B are reſpectively equal to the rectangles contained by the right lines KC, K*t*, KH and the given right line DG. Therefore KC is leſs than K*t*, and K*t* leſs than KH; ſo that the tangent E*t* muſt paſs betwixt the arches EC, EH; and, theſe arches being therefore convex towards each other, E is a cuſpis of the firſt kind.

FIG. 101. 271. *Caſe* 2. Let *e* be a cuſpis of the firſt kind, and if the tangent at *e* be not parallel to the baſe, the point E ſhall be a cuſpis of the ſecond kind. For let the perpendicular at P meet the arches *ce*, *eb*, CE, EH in the points *n*, N, *m* and M, the tangent E*t* in R, and EK, *el* parallel to the baſe in V and S. Then, by art. 256. the right lines VM, V*m* are to be taken from V towards P when N deſcribes *ce* or *n* deſcribes *eb*. The rectangles contained by VM, V*m* and VR, and by the given right line DG, are reſpectively equal to the areas D*e*NP, D*en*P and D*e*SP; and the areas D*e*NP, D*en*P are either both leſs or both greater than D*e*SP. Therefore VM, V*m* are both leſs or both greater than VR, the arches EM, E*m* are on the ſame ſide of the tangent E*t*, and, conſequently, E is a cuſpis of the ſecond kind.

272. *Caſe* 3. When *e* is a cuſpis of the ſecond kind, it appears in the ſame manner that the arches EM, E*m* are on the ſame ſide of the tangent E*t*, and therefore E is a cuſpis of the ſame kind.

273. COR. I. In the firſt caſe, the point E is a cuſpis of the firſt kind, and the right line which meaſures the fluxion of the baſe vaniſhes when that which meaſures the ſecond fluxion of the ordinate is given. The point E is alſo a cuſpis of the firſt kind when the right line which meaſures the ſecond fluxion of the ordinate DE vaniſhes, the fluxion of the baſe being gievn, if the two arches *ec*, *eb* are on the ſame ſide of DE and are convex towards each other. But E is not always a cuſpis of this kind when the right line which meaſures the ſecond fluxion of the ordinate vaniſhes, that which meaſures the fluxion of the baſe being given, or when the latter vaniſhes the former being given, though the arches CE, EH be on the ſame ſide of DE. For in this caſe *e* and E may be each a cuſpis of the ſecond kind.

274. COR.

274. Cor. II. In the second case, E is a cuspis of the second kind, and the right line which measures the third fluxion of the ordinate DE vanishes when that which measures the fluxion of the base is given, or the latter vanishes the former being given.

275. Cor. III. When the right line which measures the second fluxion of the ordinate DE is in an assignable ratio to that which measures the fluxion of the base, and the arches CE, EH are on the same side of the ordinate, then the point E is always a cuspis of the second kind, whether the third fluxion of DE be assignable or not: for in that case the tangent of the arch *ce* at *e* is oblique to D*e*, the point *e* is a cuspis either of the first or second kind; and therefore E is a cuspis of the second kind, by art. 271. and 272. When the ratio of the right line which measures the second fluxion of the ordinate DE to that which measures the fluxion of the base is not assignable, (the arches CE, EH being still on the same side of DE) if the ratio of the right line which measures the third fluxion of DE to that which measures the fluxion of the base be assignable, E is still a cuspis of the second kind; but when this latter ratio also is not assignable, E may be a cuspis of either kind. Fig. 100. & 101.

276. The ordinate DE is a *maximum* or *minimum* of the second kind in all cases when E is a cuspis of either kind, that only excepted wherein E is a cuspis of the first kind and the tangent at E is at the same time parallel to the base. And we may conclude DE to be a *maximum* or *minimum* of this kind, if the curve is not continued on both sides of the ordinate, not only when the first fluxion of the ordinate is to the fluxion of the base in an assignable ratio, but also when the fluxion of the base being given the fluxion of the ordinate of any subsequent order is to the fluxion of the base in an assignable ratio. For in all those cases the point E is a cuspis of the second kind; or, if it is a cuspis of the first kind, the tangent at E is not parallel to the base.

277. What we have said of the greatest and least ordinates is easily applied to the greatest or least rays that can be drawn from a given point to a curve. Let S be a point that is given in Fig. 102.
the plane of the curve DPE, SA a right line given in position, SP any right line from S that meets the curve in P, PM a per-

pendicular from P on SA in M. Then, the fluxion of SM being given, the ray SP is a *maximum* or *minimum* when its first fluxion vanishes and its second fluxion does not vanish, or when the fluxions of the subsequent successive orders vanish if the number of all those fluxions that vanish be an odd number; for if MQ be taken upon MP always equal to SP, the ordinate MQ of the curve IQ shall in those cases be a *maximum* or *minimum* by prop. 21. When the number of the successive fluxions of SP or MQ that vanish is an even number, the point Q is a point of contrary flexure in the curve IQ, and MQ or SP is neither a *maximum* nor *minimum*: but it does not follow, that P is a point of contrary flexure in the curve DPE.

278. To illustrate by an obvious example the necessity of having regard to those limitations in resolving by the common rules the problems that relate to the *maxima* and *minima*, let DPE be a common parabola, DK its axis, S a point given within the curve, SA be perpendicular to the axis DK in K, and PN perpendicular to it in N. Then, SM or PM being supposed to flow uniformly, let the fluxions of SP be computed. The first fluxion of SP vanishes when SP becomes perpendicular to the curve, and by the common rule SP ought in that case to be a *maximum* or *minimum*. But it is certain, that if DK be equal to the sum of 3DN and one half of the parameter of the axis, and the point S be not upon the axis, the right line SP, though perpendicular to the curve, is neither a *maximum* nor *minimum*; for a circle described from the center S through P falls without the arch PD and within the arch PE, as shall be demonstrated afterwards. In this case it will be found, that the second fluxion of SP vanishes as well as its first fluxion, but that its third fluxion does not vanish; so that, according to prop. 21. SP is not a *maximum* or *minimum* in this case, though its first fluxion vanish. If the third fluxion of SP also vanish, then SP is a *maximum* or *minimum* by the same proposition. And this is the case when the point S is upon the axis at *f* and *f*D is equal to one half of the parameter; for, in this case, when P comes to D, the first, second and third fluxions of *f*P vanish, but its fourth fluxion does not vanish, as will appear by the computation; and it is known that *f*D is the least

least right line that can be drawn from the point f to the parabola. Nor is there any curve but the circle alone that does not afford examples of this kind. It may be of use for abridging computations of this sort to observe, that when the value of any quantity z is expressed by a term that is formed from any power of another quantity u and invariable quantities, if while u is finite the first fluxion of u and its fluxions of any subsequent successive orders vanish, the fluxions of z of the same orders vanish at the same time. If z be expressed by a fraction whose numerator N and denominator D are both finite when the fluxion of N is to the fluxion of D as N is to D, then the fluxion of z vanishes; and if the fluxions of N and D of any subsequent successive orders be to each other respectively in the same ratio of N to D, then the fluxions of z of the same orders vanish.

279. Let us now resume the construction of the 211th and 212th articles, where P the intersection of LP the tangent of any curve LB and of SP the perpendicular from the given point S is supposed to be always found in the curve DPE; and where it is shewn, that if the angle SPT be made equal to SLP, PT shall be the tangent of the curve DPE. Let ST be always perpendicular from S upon this tangent PT, and T be always found in the curve GTH; and, if AS be any right line given in position that produced beyond S meets PT produced beyond T in Q, the angles ASL, ASP, AST shall be in arithmetical progression, and the angle AST shall be equal to 2ASP—ASL. The angle ASP being supposed to increase, the angle SQT increases or decreases according as the arch of the curve described by P is concave or convex towards S; and, because AST (or 2ASP—ASL) is equal to SQT added to the right angle STQ, it follows, that, the fluxion of the angle ASP being supposed positive, the arch described by P is concave or convex towards S according as the fluxion of 2ASP—ASL is positive or negative, that is, according as the fluxion of 2ASP is more or less than the fluxion of ASL. When the arch described by L is convex towards S, ASL decreases while ASP increases, the fluxion of ASL is negative, the angle AST increases, and the arch described by P is always concave towards S. These things were mentioned above, but without a proof. FIG. 103.

 280. When

FIG. 104. 280. When the point P in the curve DE is a point of contrary flexure, the point T in the curve GTt is a cuspis, the angle AST (or 2ASP—ASL) is a *maximum* or *minimum*; and the right line ST is also a *maximum* or *minimum*, unless when the angle SPT or SLP is a right one, or when S and P coincide. For let the angle STK be made equal to SPT so that TK may ly the same way from ST as PT from SP, and TK shall be the tangent of the curve GTt at T, by art. 211. & 212. Let P*p* be the arch terminated at P that is concave towards S, and Pp the arch that is convex towards it, and while P describes the arches P*p*, Pp, let T describe the arches T*t*, Tt. It is manifest, that these arches T*t*, Tt are on the same side of the right line PT; and, since they have the same tangent TK, the point T is a cuspis. The angle SQT is a *maximum* or *minimum* when P is a point of contrary flexure, and therefore AST (or 2ASP—ASL) is a *maximum* or *minimum*. When the angle STK or SPT is not a right one, the right line ST either exceeds the right lines that can be drawn from S to the adjoining parts of either arch T*t*, Tt, or is less than them, and therefore ST is a *maximum* or *minimum*.

281. It follows, conversely, that the curve *p*p being continued on both sides of the right line PL, and the tangent at P being oblique to SP, if ST be a *maximum* or *minimum*, the point P in the curve DPE is a point of contrary flexure. Therefore, *f*P an arch of a circle described from the center S through P being supposed to flow uniformly, if the fluxion of ST vanish, the point P is a point of contrary flexure, the same limitations being understood as were described in prop. 21. Let the invariable fluxion of the arch *f*P be represented by a given line PI taken upon PL, and let IH perpendicular to PI meet TP produced in H; then shall PH and IH represent the fluxions of the curve DP and ray SP by prop. 16. & 17. The rectangle contained by SP and PI is equal to the rectangle contained by ST and PH; and therefore, when the fluxion of ST vanishes, the rectangle contained by IH (which measures the fluxion of SP) and PI is equal to the rectangle contained by ST and the right line which measures the fluxion of PH, and the fluxion of PH is to the fluxion of SP as PI is to ST or as PH is to SP.

But,

But, becauſe PI is ſuppoſed to be invariable, it follows from prop. 15. that the fluxion of IH is to the fluxion of PH as PH is to IH. Therefore the fluxion of IH is to the fluxion of SP (which is expreſſed by IH) as the ſquare of PH is to the rectangle contained by SP and IH; and, conſequently, the right line which meaſures the ſecond fluxion of SP, PH which meaſures the firſt fluxion of the curve, and the ray SP are in continued proportion; which coincides with one part of the rule that is uſually given for finding the points of contrary flexure in curves, that are conſidered as ſpirals, and are defined by an equation that expreſſes the relation of the fluxions of ƒP, SP or DP to each other. But this rule is not to be admitted without the limitations that follow from prop. 21. though the firſt fluxion of ST vaniſh. If the curve deſcribed by P be not continued on both ſides of the right line PL, or if the fluxions of ST of the ſubſequent ſucceſſive orders vaniſh, and the number of all its fluxions that vaniſh be an even number, we cannot conclude that P is a point of contrary flexure. When the firſt fluxion of ST vaniſhes, its ſecond fluxion vaniſhes alſo (the fluxion of the arch ƒP being invariable) when the ratio of the ſecond fluxion of PH to the ſecond fluxion of SP is the ſame as that of PH to SP; and when the fluxions of PH of any ſucceſſive orders from the firſt, are to the fluxions of SP of the ſame orders, reſpectively, in the ſame ratio of PH to SP, then the fluxions of ST of theſe orders alſo vaniſh.

282. It follows alſo from the 280th article, that, the curve *pp* being continued on both ſides of the right line PL and the angle SPT being acute, P is a point of contrary flexure, when the fluxion of the angle AST vaniſhes, or when the fluxion of the angle ASP becomes equal to one half of the fluxion of ASL, (thoſe caſes however being excepted in which AST is not a *maximum* or *minimum* according to prop. 21.) that is, when the angular velocity of SP about S is one half of the angular velocity of SL about S, and is in the ſame direction; for if the motion of SL and SP be in different directions, the angular motion of ST does not vaniſh when the angular motion of SP is equal to one half of the angular motion of SL, but on the contrary is equal to the ſum of the motion of SL added to twice

twice the motion of SP. The point T in the curve GTH is a point of contrary flexure, for the same reason, when the fluxion of AST becomes equal to one half of the fluxion of ASP, or when the fluxion of ASP becomes equal to two thirds of the fluxion of ASL, because ASL, ASP, AST are in arithmetical progression. In general, the series of curves BL, DP, GT, &c. being continued, each of which is supposed to pass through the intersections of the tangents of the preceeding curve with the perpendiculars from S on these tangents, that point in the last curve of the series which corresponds to L in the first curve is a point of contrary flexure when the fluxion of the angle ASL is to the fluxion of ASP as the number of curves in the series is to the same number diminished by unit; and that point is a cuspis when the fluxion of ASL is to the fluxion of ASP as the number of curves is to the same number diminished by two.

283. In this last manner the points of contrary flexure and
FIG. 105. reflexion are sometimes easily discovered. For example, when ALB is a circle, C the center, S any point within the circle, AB the diameter that passes through S, the point P in the curve DPE is a point of contrary flexure when the square of LS is one third part of the rectangle ASB, or (LS being produced till it meet the circle again in Z) when LS is one fourth part of LZ. For, let CV be perpendicular on LZ in V; and, SLR being made equal to CLQ if LR meet the diameter in R, the fluxion of ASL shall be to the fluxion of ACL (or ASP) as CR is to SR (by prop. 18.) or as LV is to LS; and therefore, when LS is equal to one half of LV, or one fourth part of LZ, the fluxion of ASP is one half of the fluxion of ASL, and P is a point of contrary flexure; that case being excepted wherein CS is equal to SB, and SB is one fourth part of AB, in which the point L coincides with B when the fluxion of ASP becomes equal to one half of the fluxion of ASL, and the curve described by P is not continued on both sides of the tangent at B. When S is nearer to the center of the circle than to the circumference, or when S is without the circle, or upon the circumference, the fluxion of ASP never becomes equal to one half of the fluxion of ASL. When S is without the circle, the angular velocity of SP may become equal to one half of the angular velocity

of

of SL; but the motions of SP and SL are in oppoſite directions when this happens, the fluxion of ASP being ſuppoſed poſitive the fluxion of ASL is negative, and the fluxion of AST does not vaniſh. The curve ALB being ſtill a circle, and the ſeries of curves ALB, DPE, GTH, &c. being continued, the point in the laſt curve of the ſeries that correſponds to L in the circle is a point of contrary flexure when SL is to SZ as the number of curves diminiſhed by unit is to the ſame number increaſed by unit, or when the ſquare of SL is to the given rectangle ASB in that ratio; and that point is a cuſpis when SZ is to SL as the number of curves is to this number diminiſhed by two; the caſe being always excepted when S is ſo ſituated that the point L coincides with B when this happens.

284. In like manner theſe points are readily determined in many other curves, eſpecially when the curve BL is ſuch that a point can be aſſigned, from which rays being drawn to the curve and perpendiculars to the tangents of the curve, the angular velocity of the ray about that point is to the angular velocity of the perpendicular in any invariable ratio: When BL is any curve of this kind, it has no point of contrary flexure; and if S coincide with that given point, the curves DPE, GTH being of the ſame kind *, have alſo no point of contrary flexure; but if S be any other point in the plane of the curve BL, the curves DPE, GTH, &c. may have points of contrary flexure and reflexion that are often eaſily determined by art. 282. & 210. When BL is a parabola and S is within the parabola upon the axis, P is a point of contrary flexure when LQ is equal to LS, or when BM (LM being perpendicular to the axis in M) is one third part of BS. The curve deſcribed by T in this caſe has two points of reflexion correſponding to the two points of contrary flexure in the curve DP, and a third point of reflexion upon the axis of the parabola at a diſtance from S towards A equal to one fourth part of the parameter. FIG. 106.

285. The right line ST is alſo a *maximum* or *minimum*, if the fluxion of fP vaniſh when the fluxion of ST is ſuppoſed to be aſſignable, by art. 245. and in this caſe the ratio of the difference betwixt the rectangle PHI and the rectangle contained by FIG. 104.

* Deſcript. curvarum, part. 2. prop. 14. 16. &c.

SP

SP and the right line which meaſures the fluxion of PH to the ſquare of PH by increaſing becomes unaſſignable; and this coincides with the ſecond part of the rule that is commonly given for finding the points of contrary flexure in curves that are conſidered as ſpirals. It is however to be allowed with limitations analogous to thoſe above mentioned in the 245th, 262d and 267th articles. But having inſiſted on this ſubject at a ſufficient length, we proceed to conſider ſome other affections of curve lines.

CHAP. X.

Of the Aſymptotes of curve Lines, the Areas bounded by them and the Curves, the Solids generated by thoſe Areas, of ſpiral Lines, and of the Limits of the Sums of Progreſſions.

286. A Right line given in poſition is an *Aſymptote* of the branch of a curve when they never meet, but approach to each other continually, ſo that by producing them their diſtance from each other becomes equal to any line how ſmall ſoever that may be given; and the branch of the curve that approaches thus to the aſymptote is ſaid to be of the *hyperbolic* kind. A branch of a curve that approaches in the ſame manner to a parabola is ſaid to be of the *parabolic* kind and to have a parabolic aſymptote, of which there may be as many different kinds as there are parabolas of different orders. In general, any curve lines may be ſaid to be aſymptotes of each other mutually when they approach to each other in this manner.

287. In the common doctrine of the hyperbola it is ſhewn, that a curve and a right line may continually approach to each
Fig. 107. other in this manner and never meet. Let CO and CV be the right lines that are called the aſymptotes of the hyperbola *a*ME, *a*K and IM two right lines parallel to CV bounded by the aſymptote CO and the curve in *a* and K, I and M; and IM ſhall

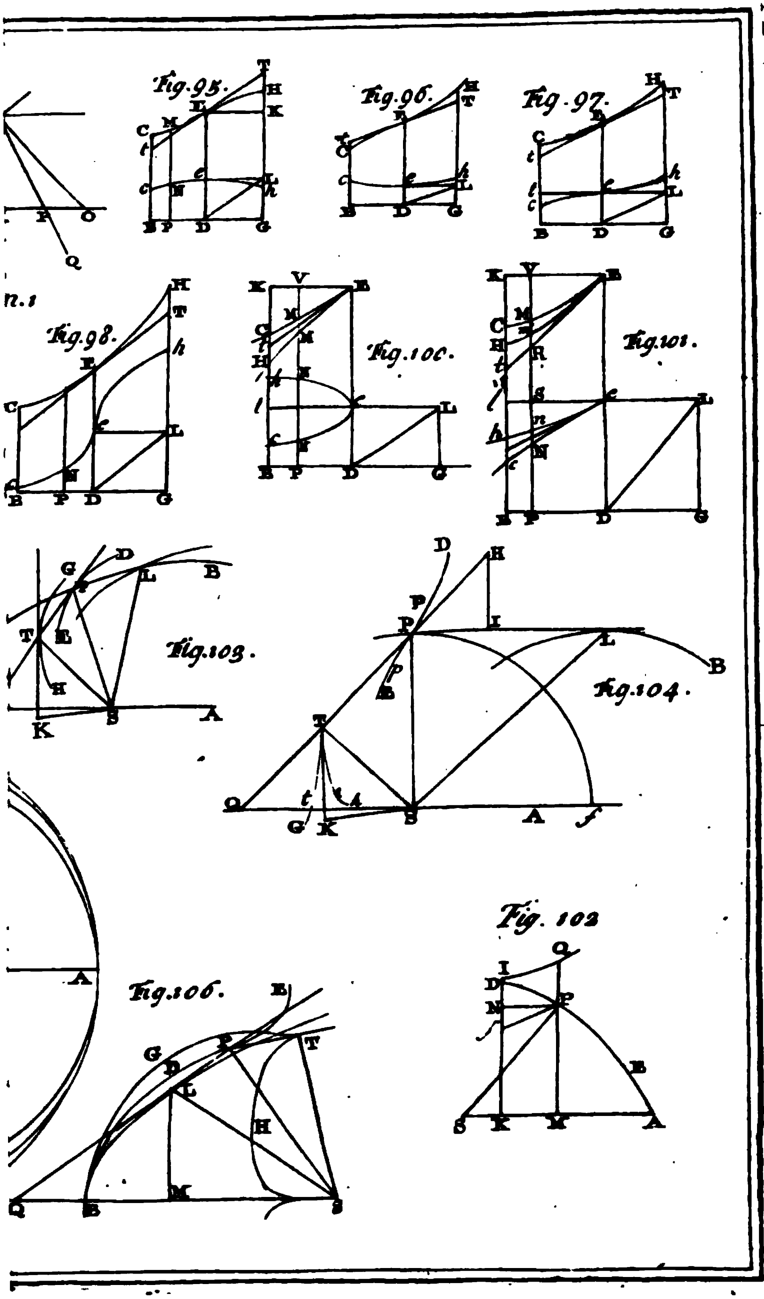


Fig. 95.
Fig. 96.
Fig. 97.
Fig. 98.
Fig. 100.
Fig. 101.
Fig. 103.
Fig. 104.
Fig. 102
Fig. 106.

shall be to aK as CK is to CI. Therefore, CK and aK being given, and CI being produced till it become any multiple how great soever of CK, the ordinate IM is always assignable, being the same part of aK as CK is of CI; and IM continually decreases, so that it may become less than any given line Z by producing CI till its ratio to CK be greater than that of aK to Z. When the ordinate IM is to aK as the square, cube, or any power of CK that has a positive number for its exponent, is to the same power of the absciss CI, the curve aME is of the hyperbolic kind, and the same right lines CO, CV are its asymptotes.

288. Let AD be a right line given in position, S a given point, SPM any right line from S meeting AD in P; upon which let PM be taken always from P equal to a given line Aa: then the curve aME shall be the *conchoid* of the ancients, and the right line AD shall be its asymptote. For the curve aM shall never meet this right line AD, because PM is supposed to be taken always equal to the given line Aa; and as it never decreases, so it cannot be supposed to vanish. But the curve continually approaches to AD, and MN the perpendicular upon AD may become equal to any right line Z how small soever that may be given, by producing the figure; for if AQ be taken upon Aa equal to Z, Qm parallel to AD meet the circle amx described from the center A in m, and SPM be drawn parallel to Am meeting Qm produced in M, the point M shall be in the conchoid, and MN equal to AQ shall be equal to Z. FIG. 108.

289. To mention one other simple instance: Whoever admits that a right line may be continued at pleasure, and that any given right line may be divided into two equal right lines, (according to the principles of the common geometry,) will allow, that the base of a figure being produced, the ordinate may be conceived to decrease in such a manner, that when the base is increased by any given line, the ordinate may become one half of what it was before the base acquired that increment; in which case the ordinate never vanishes, because the half of a right line is always a right line. Let OA, AD, DE, EF, &c. be any equal right lines by which the base is successively increased; let Aa the ordinate at A be one half of Oo the ordi- FIG. 44.

nate at O, D*d* one half of A*a*, E*e* one half of D*d*, and so on. And it is manifest, that the logarithmic curve (described in art. 176.) which passes through the points *o*, *a*, *d*, *e*, *f*, and the extremities of all the ordinates in the same series, which may be continued at pleasure, can never meet the base. In general, any quantity may be conceived to decrease continually and yet never be quite exhausted; as, when the right line AP that touches the circle A*gx* in A is produced and KP is always joined, the arch *xg*, the angle *x*K*g*, and the perpendicular *gf* decrease continually, so that they may become less than any given quantity of the same kind by producing AP to an assignable distance, but never vanish.

FIG. 107.

290. We have mentioned those simple instances, to shew that there is nothing so abstruse or inconceivable in what Geometricians demonstrate concerning the asymptotes of curve lines as is sometimes represented. They are under no necessity of supposing, that * *a finite quantity or extension consists of parts infinite in number*, or that there are any more parts in a given magnitude than they can conceive and express: It is sufficient that it may be conceived to be divided into a number of parts equal to any given or proposed number, and this is all that is supposed in strict geometry concerning the divisibility of magnitude. It is true, that the number of parts into which a given magnitude may be conceived to be divided is not to be fixed or limited, because no given number is so great but a greater than it may be conceived and assigned: but there is not therefore any necessity for supposing that number infinite; and if some may have drawn very abstruse consequences from such suppositions, geometry is not to be loaded with them.

291. Though Geometricians are under no necessity of supposing a given magnitude to be divided into an infinite number of parts, or to be made up of infinitesimals, they cannot so well avoid the supposing it to be divided into a greater number of parts than may be distinguished in it by sense in any particular determinate circumstances. But they find no difficulty in conceiving this; and such a supposition does not appear to be repugnant to

* Princip. of human knowledge, § 124.

the

the common ſenſe of mankind, but on the contrary to be moſt agreeable to it, and to be illuſtrated by common obſervation. It would ſeem very unaccountable, not to allow them to conceive a given line, of an inch in length for example, viewed at the diſtance of ten feet, to be divided into more parts than are diſcerned in it at that diſtance; ſince by bringing it nearer a greater number of parts is actually perceived in it. Nor is it eaſy to limit the number of parts that may be perceived in it when it is brought near to the eye, and is ſeen through a little hole in a thin plate, or when by any other contrivance it is rendred diſtinct at ſmall diſtances from the eye. If we conceive a given line that is the object of ſight to be divided into more parts than we perceive in it, it would ſeem that no good reaſon can be aſſigned why we may not conceive tangible magnitude to be divided into more parts than are perceived in it by the touch, or a line of any kind to be divided into any given number of parts, whether ſo many parts be actually diſtinguiſhed by ſenſe, or not. If the hyperbola and its aſymptote were accurately deſcribed, they would ſeem to ſenſe to join each other, at various diſtances from the center according to the different circumſtances in which they might be perceived; but we may conceive the ordinate at the point where they ſeem to join to have a real magnitude, in the ſame manner as we conceive a given line to ſubſiſt when it is carried to ſo great a diſtance that it vaniſhes to ſight, or any ſmall particle (as an atom in the ſun-beams) to exiſt, though it eſcape the touch, or have no tangible magnitude. It may perhaps illuſtrate this, if it be conſidered, that the curve cannot be ſaid to meet its aſymptote in this caſe, in the ſame ſenſe that a circle is ſaid to meet its diameter, which it appears to interſect in all caſes, whatever the diſtance or poſition of the figure, or the acuteneſs of the ſenſe may be; whereas the ordinate of the hyperbola that vaniſhes to ſight at a greater diſtance, becomes viſible at a leſs diſtance, and may be diſtinguiſhed into more and more viſible parts, in proportion as it approaches to the eye, or the ſenſe is more acute. And ſurely it muſt be allowed that there is ground for a difference between a line that eſcapes the ſight and vaniſhes, becauſe of its diſtance from the eye, and a

line that in no case can ever be perceived, or can be supposed to have any existence. Perhaps it will be said by some, that, strictly speaking, it is not the same line that in those different circumstances has a greater and less number of visible parts. In answer to this, it is sufficient for our purpose to observe, that as there can hardly be any Philosopher but will allow that there is some sense in which it is the same inch-line that has more visible parts at eight inches distance from the eye than when it is held at the length of the arm; so it is not incumbent on us to explain in what sense this is to be understood according to every scheme: It is enough that this sense must be supposed to be plain and obvious, as it is universal, and that Geometricians ought to be allowed to consider lines and figures in this sense as well as every body else. Philosophers and the vulgar equally conceive the sun and planets, and the other objects of their observation and enquiries, to be the same bodies, when seen at different distances or different times: And if they were not allowed to consider those bodies as made up of more parts than are perceived by sense, and Geometricians were under the same limitations as to magnitude in general, they would not be a little perplexed; nor is it the more intricate and subtile part of those sciences only that would be thus pared off. The learned author abovementioned tells us, "That the magnitude of the object which "exists without the mind, and is at a distance, continues always invariably the same *." He seems to speak of tangible magnitude. It is not our business here to enquire how, according to his doctrine, tangible magnitude can be conceived to exist without the mind any more than visible magnitude. This concession perhaps is made only for the sake of his Argument in this place; but the evidence for the existence of such an object may very well be supposed to approach to that which we have for the existence of any other objects that are not immediately perceived by us. And since he admits it, and argues from it, in this treatise, it would seem that some invariable magnitude is to be allowed, which † we apprehend by

* New Theory of Vision, § 55.
† Ibid. § 54.

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the ſight, though not immediately; and that this magnitude may be conceived to be divided into any given number of parts, from the demonſtrations propoſed by Geometricians on this ſubject. In applying which, it ought to be remembred, that a ſurface is not conſidered by them as a body of the leaſt ſenſible magnitude, but as the termination or boundary of a body; a line is not conſidered as a ſurface of the leaſt ſenſible breadth, but as the termination or limit of a ſurface: nor is a point conſidered as the leaſt ſenſible line, or a moment as the leaſt perceptible time; but a point as a termination of a line, and a moment as a termination or limit of time. In this ſenſe they conceive clearly what a ſurface, line, point *, and a moment of time is; and the *poſtulata* of Euclid being allowed and applied in this ſenſe, the proofs by which it is ſhewn that a given magnitude may be conceived to be divided into any given number of parts, appear ſatisfactory: And if we avoid the ſuppoſing the parts of a given magnitude to be infinitely ſmall, or to be infinite in number, this ſeems to be all that the moſt ſcrupulous can require.

292. But to proceed: The arch A*g*, the ſector AK*g*, and the ordinate PM increaſe continually while AP is produced; but the arch A*g* never amounts to A*x*, the ſector AK*g* to the ſector AK*x*, or the ordinate PM to the given right line AK the diſtance betwixt the baſe AP and aſymptote KI. An area APNF may increaſe at the ſame rate as the arch A*g*, the ſector AK*g*, or the ordinate PM increaſes; and, by flowing in the ſame manner, it may approach in magnitude to a given ſpace continually while the figure is produced, and never amount to it. Let us reſume, for an example, the conſtruction of art. 289. and, O*o* being biſected in M, OM in N, ON in R, and ſo on, let the rectangles O*o*BA, A*a*HD, D*d*IE, E*e*KF, &c. be completed, and let N*d*, R*e* meet A*a* in L and V: The rectangles AH, DI, EK, &c. are reſpectively equal to the rectangles *oa*, ML, NV, &c. and the ſum of thoſe rectangles being ſucceſſively equal to *oa*, *o*L, *o*V, &c. it is therefore always leſs than the rectangle *o*A, though the ſeries of thoſe rectangles be conti- Fig. 107. Fig. 44.

* The Analyſt, § 31.

nued.

nued till their number become equal to any given number how great soever it may be. The differences betwixt their successive sums and the rectangle *o*A are successively equal to the rectangles AM, AN, AR, &c. which continually decrease, and by continuing the series may become less than any given space. Therefore the rectangle *o*A is the limit to which the sum of the rectangles AH, DI, EK, &c. continually approaches while the figure is produced, and to which it never amounts unless the figure be supposed to be infinitely produced. As for the area bounded by the curve *adef* and the base AF, it is always less than the sum of the rectangles AH, DI, EK, &c. and therefore is always less than the rectangle *o*A. It approaches to a lesser limit, *viz.* the rectangle contained by A*a* and O*o*, if we suppose *o* to be the point where the tangent makes an angle with the ordinate that is half a right one; and the limit of the area bounded by the curve *oadef*, ordinate O*o* and base OF is the square of O*o*. For let any ordinate P*p* meet *o*B in Z, and the fluxion of P*p* (or of *p*Z) shall be to the fluxion of the base OP as P*p* is to O*o*, by art. 176. Therefore the fluxion of the area OP*po* is equal to the fluxion of the rectangle contained by O*o* and *p*Z, by prop. 4. and the area OP*po* is equal to this rectangle by art. 24. And as *p*Z is always less than O*o*, but approaches to it so that the difference P*p* may become less than any given line by producing the figure, so the area OP*po* is always less than the square of O*o*, but approaches to it continually, so that their difference (the rectangle contained by P*p* and O*o*) may become less than any given space.

FIG. 107. 293. The right line KI parallel to the base AP being the asymptote of any curve *a*ME, let PN the ordinate from the curve FN*e* be always to the given right line DG as the fluxion of the ordinate PM is to the fluxion of the base, and let PM meet *ap* parallel to the base in *p*. Then the base AP shall be the asymptote of the curve FN*e*, and the area APNF shall be always less than the rectangle contained by K*a* and DG, though the base AP and curve FN be produced to any distance how great soever, but shall continually approach to that rectangle, so that their difference may become less than any given space by producing the figure. For it appears, as in art. 248. that the area APNF

APNF is always equal to the rectangle contained by pM and the given right line DG, and therefore is always less than the rectangle contained by Ka and DG by a space equal to the rectangle contained by IM and DG, which may become less than any given space, but never vanishes; because IM may become less than any given right line, but never vanishes, since KI is supposed to be the asymptote of the curve aME.

294. When CV perpendicular to CO is also an asymptote of the curve EMa produced beyond a, the right line Cv (which is the continuation of CV) is an asymptote of the curve described by N continued on the other side of AF; and, if the ordinates PM, PN meet the base on the other side of A, the area APNF shall be still equal to the rectangle contained by pM and DG, and will in this case exceed any given space by continuing the curve FN alongst the asymptote Cv, because pM will exceed any given right line by continuing the curve aM alongst the asymptote CV. In the former case, when the curve FN was produced alongst the base AD, which is one of its asymptotes, the area APNF was always less than a certain space, (the rectangle contained by Ka and DG,) which we therefore call its limit. In the latter case, when the curve FN is produced alongst its other asymptote Cv, the area APNF may exceed any given space, and has no assignable limit. They who scruple not to suppose the curve and asymptote to be infinitely produced, say, that the area APNF then becomes equal to its limit in the former case, and that it becomes infinitely great in the latter case. And this area has been said in certain cases to be more than infinite by some authors, from an analogy they imagined to be betwixt what is negative, nothing, and finite, and what is finite, infinite, and more than infinite.

295. But when the curve EMa continued on the other side of A touches the right line CV in L, the right line Cv is still an asymptote of the curve described by N continued on the other side of F, because the ratio of the fluxion of PM to the fluxion of the base, or of PN to DG, may exceed any given ratio while P describes AB; but in this case the area APNF (which is equal to the rectangle contained by pM and DG) is always less than a certain assignable space, *viz.* the rectangle Fig. 109.

con-

contained by bL and DG, (pa being supposed to meet CV in b,) because pM is always less than bL. In this case, the curve FN being continued alongst both the asymptotes BD, Bv at pleasure, the area bounded by the two ordinates PN, PN that are on different sides of AF, is always less than the rectangle contained by CL and DG, to which it approaches however as its limit when the figure is produced continually both ways. And when the curve is supposed to be infinitely produced both ways, and the asymptotes BD, Bv are also supposed to be both infinitely produced, the whole area included betwixt the curve and its two asymptotes is said to become equal to this rectangle contained by CL and DG.

FIG. 107. & 109. 296. While the point N describes the branch of the curve FN that proceeds alongst the base AP, which is one of its asymptotes, if Pn be to DG as the fluxion of PN is to the fluxion of the base AP, the area $APnf$ shall be always equal to the rectangle contained by DG and the excess of AF above PN; and the rectangle contained by AF and DG is the limit to which the area $APnf$ continually approaches by producing the curve fn and base AP. And if a series of curves be deduced in this manner, (the ordinate of any curve being to the given right line DG as the fluxion of the ordinate of the preceeding curve in the series is to the fluxion of the base,) the rectangle contained by the ordinate at A of any curve of the series and by the right line DG shall be the limit to which the area of the subsequent curve continually approaches while it is produced alongst the base. While the point N describes the branch that proceeds alongst the asymptote Bv, the point n describes a curve that has the same asymptote; but the area $APnf$ in this case may exceed any given space by producing the curve fn, and has no assignable limit: and the same is to be said of the area of any subsequent curve in the series.

FIG. 110. 297. When the curve aME has not an asymptote parallel to the base AP, but the angle kMT formed by MT the tangent at M and Mk parallel to the base decreases so that by producing the curve it may become less than any given rectilineal angle, but never vanishes, the base AP is still an asymptote of the curve described by N, because the ratio of PN to DG (which is the

same

ſame as that of kT to Mk by prop. 14.) may become leſs than any given ratio, and yet never vaniſhes. In this caſe the area APNF may exceed any given ſpace by producing the curve and baſe, becauſe pM may exceed any given right line. When CV is an aſymptote of the curve EMa produced on the other ſide of a, Cv is an aſymptote of the curve deſcribed by N produced on the other ſide of F; and, PN being on the other ſide of AF, the area APNF may exceed in this caſe alſo any given ſpace by producing the curve along the aſymptote Cv. Thus, when aME is the logarithmic curve deſcribed in art. 176. bV its aſymptote, ba the ordinate whoſe logarithm is nothing, the curve FN is the common hyperbola, becauſe the fluxion of PM is to the fluxion of BP (or bp) as ba is to BP, (by art. 176.) and, PN being to DG in the ſame ratio, the rectangle BPN is equal to the invariable rectangle contained by ba and DG. It follows from the geneſis of the logarithmic, that pM may exceed any given right line by producing the curve on either ſide of Aa, and therefore the area APNF may exceed any given ſpace by producing the hyperbola on either ſide of AF: The ſame appears from the common doctrine of the hyperbola. The curve fn (art. 293.) in this caſe, and all the ſubſequent curves of the ſeries, are hyperbolas of an higher order; and when P is upon the ſame ſide of A with D, the area terminated by the curve and baſe and the ordinates at A and P has an aſſignable limit which it can never exceed, the ſame that was defined in the laſt article; but when P is on the other ſide of A, betwixt A and B, that area may exceed any given ſpace by producing the curve.

298. The converſe of the 293d article eaſily appears. Let the baſe AD be an aſymptote of the curve FN, and ap parallel to the baſe through any given point in the perpendicular at A meet PN in p; let the rectangle contained by pM and the given right line DG be always equal to the area APNF; let the rectangle ad (Kd being equal to DG) be the limit to which the area APNF continually approaches ſo that their difference becomes leſs than any given ſpace by producing the figure, if any ſuch limit is aſſignable; and let aK be taken on the ſame ſide of ap with pM: then a right line through K parallel to the FIG. 107.

base shall be the asymptote of the curve *a*ME. For the rectangle contained by *p*M and DG shall increase in the same manner as the area APNF (to which it is always equal) increases: and as this area approaches continually to the rectangle contained by K*a* and DG in such a manner that their difference becomes less than any given space by producing the figure APNF; so the ordinate *p*M approaches continually to *a*K or *p*I, and their difference IM becomes less than any given right line by producing the figure *a*MIK. Therefore KI is the asymptote of the curve *a*ME. If no such limit of the area APNF can be assigned, but this area may exceed any given space, then the branch *a*ME is not of the hyperbolic kind, and has not a rectilineal asymptote. For in this case *p*M may exceed any given right line by producing the figure *a*M*p*; and, (*kt* parallel to *p*M being supposed to meet the tangent M*t* in *t* and M*k* equal and parallel to DG in *k*,) since PN, or *kt*, may become less than any given right line, the angle *k*M*t* may become less than any given rectilineal angle by producing the curve.

299. When the ordinate PN is reciprocally as any power of BP whose index is greater than unit, a limit of the area APNF can be assigned, and the curve *a*ME has an asymptote parallel to the base. If the ordinate PN be reciprocally as the square of BP, and the curve FN with the base AP be supposed to be infinitely produced, the ordinate PN at an infinite distance is said to be an infinitesimal of the second order, because PN is to AF as the square of BA is to the square of BP: And the element of the base AP being supposed an infinitesimal of the first order, then since the element of PM is to the element of AP as PN is to DG, it follows (according to the doctrine of infinitesimals) that the element of PM in this case becomes an infinitesimal of the third order. If PN be reciprocally as any higher power of BP, the element of PM becomes an infinitesimal of an order still lower, AP being still supposed infinite: And it is proposed as a rule *, That, the base being infinitely produced, if the element of the ordinate becomes an infinitesimal two or more degrees beneath the element of the base, then we may conclude

* Geometr. de l'infini, § 950. &c.

that

that the curve has an aſymptote parallel to the baſe. But, ſince it is acknowledged that we may be led into miſtakes by this rule, unleſs regard be had to the ratio of PM to AP when AP is ſuppoſed infinite, (as when *a*ME is a cubic parabola perpendicular to *ap* in *a*, and the ordinate is ſuppoſed infinite, its element is an infiniteſimal two degrees beneath that of the baſe, becauſe in this curve the element of *p*M is to the element of *ap* as a given ſquare is to the triple ſquare of *p*M, and yet this curve has no aſymptote;) and when this ratio is known, it may be diſcovered eaſily from thence if the curve has an aſymptote parallel to the baſe, we ſhall not inſiſt on this rule further. FIG. 110.

300. Let KO parallel to the baſe be now an aſymptote of the curve FN*e*, and the rectangle contained by PM and DG be always equal to the area APNF, as in prop. 20. Let the rectangle KR be equal to the limit to which the area FNIK continually approaches while the curve FN and aſymptote KI are produced, if any ſuch limit can be aſſigned; and let AR be taken from A towards D when the curve FN*e* is betwixt the baſe AD and aſymptote KI, but in a contrary direction when the aſymptote KI is betwixt the baſe and curve FN*e*. From R towards D take R*d* equal to DG; and, *db* being parallel to PN on the ſame ſide of the baſe with PN, and equal to AK, join R*b* and it ſhall be the aſymptote of the curve AME. For, if PN meet R*b* in S, PS ſhall be to *db* (or PI) as RP is to R*d* or DG, and the rectangle contained by PS and DG equal to the rectangle RI. Therefore the rectangle contained by MS and DG is equal to the difference betwixt the area APNF and the rectangle RI, or to the difference betwixt the rectangle RK and the area FNIK. But this difference decreaſes continually while the figure is produced, and may become leſs than any given ſpace, by the ſuppoſition. Therefore MS may become leſs than any given right line; and RS is an aſymptote of the curve AME. But if the area FNIK may exceed any given ſpace by producing the curve and aſymptote, then AME is not a branch of the hyperbolic kind, and no right line can be aſſigned for its aſymptote. The angle however which the tangent at M forms with the ordinate PM approaches continually to the angle FIG. 111.

KA*k*, K*k* being taken upon KI equal to DG. And we are not to conclude that the curve has always an asymptote, when, according to the language of those who employ infinites and infinitesimals in this doctrine, the curve has an infinite branch that at its termination becomes oblique to the base.

301. It is manifest, conversely, that if RS be an asymptote of the curve AME, PN be to the given right line DG as the fluxion of PM is to the fluxion of AP; and, R*d* equal to DG being taken from R in the same direction as P is from A, if *db* parallel to PM meet RS in *b*, then a right line through *b* parallel to the base shall be the asymptote of the curve FN*e*; and the rectangle RK shall be the limit of the area FKIN, being always greater than this area by an excess that decreases continually and becomes less than any given space by producing the figure.

302. The continuation of those theorems will appear from the following proposition and its converse.

PROP. XXIV.

FIG. 112. *Let the line* B*m be an asymptote of any kind of the curve* FM; *the fluxion of the base being represented by the given right line* DG, *let the fluxions of the ordinates* PM, P*m be always measured by the right lines* PN, P*n*, *and, when* FM *is equal to* AF, *let* P*m*, PN *and* P*n be equal to* AB, A*f and* A*b*, *respectively: Then the rectangle contained by* BF *and* DG *shall be the limit of the area* b*f*N*n*.

The curve B*m* is an asymptote of the curve FM (art. 286.) when M*m* continually decreases so that it becomes equal to any right line how small soever that may be given by producing the figure but never vanishes. Let FK, BL, *fk* and *bl* parallel to the base meet PM in K, L, *k* and *l*, respectively. Because PN is to DG as the fluxion of the ordinate PM is to the fluxion of the base AP, it follows, that the fluxion of the area APN*f* is equal to the fluxion of the rectangle contained by DG and PM. There-

Therefore the area APNf is equal to the rectangle contained by DG and KM, by theor. 4. since AP and KM begin to be generated at the same time. In the same manner, the area APnb is equal to the rectangle contained by DG and Lm. And, consequently, the area bfNn (the difference of APnb and APNf) is equal to the rectangle contained by DG and the difference betwixt Lm and KM or the excess of BF above Mm. Therefore the area bfNn is always less than the rectangle contained by DG and BF; but it continually approaches to this rectangle as its limit, since Mm continually decreases and becomes less than any given right line by producing the figure. We have supposed the lines Bm and FM to be on the same side of AP; but the demonstration is easily adapted to the other cases. It is obvious however that this proposition cannot be extended to the case when the point m is found in a rectilineal asymptote of the curve FM that is perpendicular to the base; for AF and PM never meet this asymptote, being parallel to it.

303. Cor. I. Since Mm the difference of PM and Pm continually decreases and becomes less than any given right line by producing the figure, but never vanishes, it follows, that Nn which measures the difference of their fluxions continually decreases, and may become less than the given right line DG which measures the invariable fluxion of the base in any given ratio: And therefore bn is an asymptote of the curve fN.

304. Cor. II. The converse of this proposition easily appears: That if bn be an asymptote of the curve fN of any kind, and, the point F being taken any where upon Af the perpendicular to the base at A, BF be taken from F the contrary way that bf is from f, so that the rectangle contained by DG and BF be the limit of the area bfNn; and if the rectangle contained by KM and DG be always equal to the area APNf, and the rectangle contained by Lm and DG equal to the area APnb: then the curves Bm and FM shall be asymptotes to each other mutually. For the rectangle contained by Mm and DG shall be equal to the excess of the rectangle contained by BF and DG above the area bfNn; and, since this excess may become less than any given space by continuing the figure, it follows, that Mm may become less than any given right line.

But

But if the area *bf*N*n* may exceed any given ſpace by producing the figure, then B*m* is not an aſymptote of the curve FM.

305. Cor. III. When B*m* is a right line parallel to the baſe, P*n* vaniſhes, and *bn* coincides with the baſe AP, which therefore is the aſymptote of *f*N, as in art. 293. When B*m* is a right line oblique to the baſe, P*n* is invariable, and *bn* is a right

Fig. 113. line parallel to the baſe, as in art 301. When B*m* is a common parabola that has its axis perpendicular to the baſe, *bn* is a right line oblique to the baſe, and is the aſymptote of the curve *f*N: And if the right line *bn* be given in poſition, and the limit of the area *bf*N*n* be known, the parabolic aſymptote B*m* is determined by taking BF ſo that the rectangle contained by it and DG may be equal to that limit, producing *nb* till it meet the baſe in R, and upon RI parallel and equal to AB taking IE from I the ſame way as A is from *b*, ſo that the rectangle contained by IE and DG may be equal to the triangle AR*b*; for E ſhall be the vertex of the parabola required: and if RV be taken upon RI equal to 2DG and V*b* parallel to the baſe meet R*b* in *b*, then V*b* ſhall be the parameter of the axis of the parabola.

306. Cor. IV. When B*m* is a cubic parabola, *bn* is a common parabola; and when B*m* is ſuch a parabola that the ordinate P*m* is as a power of the abſciſs whoſe index is *m*, then P*n* is as a power of the abſciſs whoſe index is leſs than *m* by unit; and when *m* is leſs than unit, *bn* is ſome hyperbola that has the baſe for its aſymptote. But we have inſiſted on this ſubject ſufficiently.

307. As a right line, or area, may increaſe continually and never amount to a given right line, or rectangle; ſo a ſolid may increaſe continually and never amount to a given cube or cylin-

Fig. 114. der. Let KI parallel to the baſe AP be the aſymptote of any curve *a*ME, as in art. 293. and, K*d* being taken upon KI equal to any given right line DG, let *d*Z parallel to KA meet the baſe in Z and M*x* parallel to the baſe in *g*, and complete the rectangle *augx*. Let the ſquare of PN (the ordinate from the curve FN*e*) be always to the ſquare of DG as the fluxion of PM (the ordinate of the curve *a*ME) is to the fluxion of the baſe AP. Then the ſolid generated by the hyperbolic area

APNF

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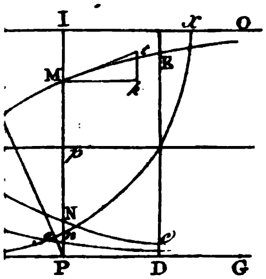
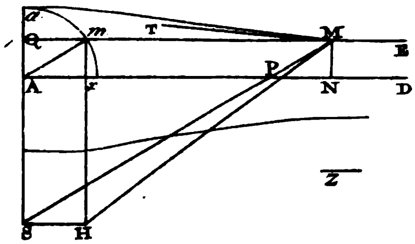


Fig. 108.



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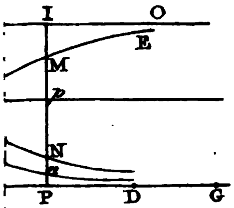

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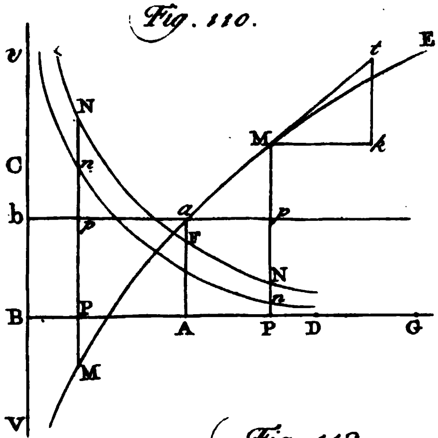


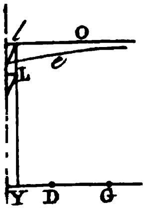


Fig. 112.

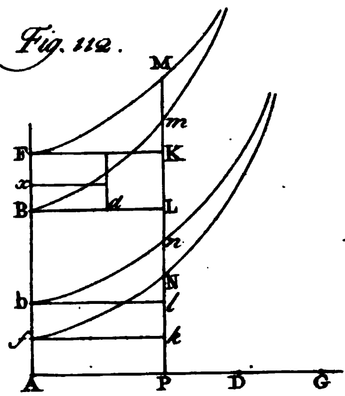
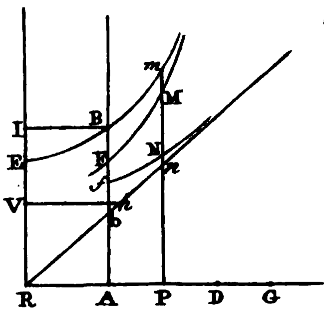


Fig. 113.



APNF revolving about the asymptote AP shall be equal to the cylinder generated by the rectangle ag revolving about the axis aK; and the cylinder generated by the rectangle ad about the same axis aK shall be the limit to which the solid generated by the area APNF shall continually approach while the figure is produced. For, since the square of PN is to the square of DG (or gx) as the fluxion of PM (or of ax) is to the fluxion of AP, a cylinder upon a circle of the radius PN of a height that measures the fluxion of the base AP is equal to a cylinder upon a circle of the radius gx of a height that measures the fluxion of ax. Therefore the fluxions of the solids generated by the area APNF about AP, and by the rectangle ag about aK, are equal, by prop. 6. and these solids themselves are equal by theor. 4. The right line ax is always less than aK, but approaches to it continually, so that their difference IM becomes less than any given right line by producing the figure; and therefore the solid generated by the area APNF about the asymptote AP is always less than the cylinder generated by the rectangle ad about aK, but approaches to it continually as its limit, so that their difference (the cylinder generated by the rectangle Kg about Kx) may become less than any given solid by producing the figure.

308. Let the square of Pn (the ordinate of the curve fn) be to the square of DG as the fluxion of PN is to the fluxion of AP, and the cylinder generated by the rectangle FZ about FA shall always exceed the solid generated by the area APnf about AP, and be equal to the limit to which this solid approaches continually while the figure is produced; for if nq parallel to the base meet dZ in q, the solid generated by APnf about the axis AP shall be always equal to the cylinder generated by the rectangle Fq about FA. If this series of curves be continued in the same manner, suppose the rectangle that has the ordinate at A of any curve in the series for its base and its height equal to Az (or DG) to revolve about AK, and the cylinder generated by it shall be the limit of the solid generated by the area of the next curve in the series revolving about the asymptote AP.

309. If the curve EMa produced on the other side of a touch FIG. 115.

BV

BV in L, let LR parallel to AP meet KA and dZ produced in k and R, and the solid generated by APNF about AP when P is betwixt A and B shall be always less than the cylinder generated
FIG. 114. by the rectangle aR revolving about the axis ak. But if BV is an asymptote of the curve EMa, the solid generated by APNF in this case may exceed any given solid by producing the figure.

310. When aME is a common hyperbola, CO and CV its asymptotes, the curve FNc is also a common hyperbola, and BA and BC its asymptotes. For in this case the fluxion of PM is to the fluxion of AP (or the fluxion of IM to the fluxion of CI) as IM is to CI, or as the rectangle CKa to the square of CI; and, by the supposition, the square of PN is to the square of DG as the fluxion of PM is to the fluxion of AP: Therefore PN is to DG as a mean proportional betwixt CK and Ka is to BP. From which it follows, that the solid generated by the hyperbolic area APNF about AP is always equal to the cylinder generated by the rectangle ag about the axis aK, and never amounts to the cylinder generated by the rectangle ad about aK, or BF about BA, but approaches to it as its limit. But if AP be taken from A towards B, the solid generated by APNF may exceed any given solid by producing the figure. It appears in the same manner, that if PN be reciprocally as any power of BP whose index is any number that exceeds $\frac{1}{2}$, then a cylinder may be assigned that always exceeds the solid generated by APNF about the axis AP, the figure being produced
FIG. 116. to any distance how great soever. When aME is the logarithmic curve and KO its asymptote, FNe is also a logarithmic and AP its asymptote. For, since the square of PN is to the square of DG as the fluxion of PM is to the fluxion of AP, or as IM to aK, (supposing aK to be the ordinate whose logarithm vanishes and is equal to the invariable subtangent of the curve;) it follows, that the fluxion of PN is to the fluxion of IM as PN is to 2IM. Therefore the fluxion of PN is to the fluxion of AP as PN is to 2aK. Hence the cylinder that is the limit of the solid generated by the area APNF in this case has its axis aK equal to one half of AT the subtangent, and the radius of the base Kd equal to the ordinate AF. The continuation of the theorems of this kind will appear from the following proposition.

PROP.

PROP. XXV.

311. *The rest remaining as in the last proposition, suppose now that the square of* PN *is to the square of* DG *as the fluxion of* PM *is to the fluxion of* AP, *and the square of* P*n* *to the square of* DG *as the fluxion of* P*m* *is to the fluxion of* AP : *Then, if the area bf*N*n* *be supposed to revolve about the axis* AP, *the solid generated by it shall be always less than a cylinder upon a circle described with the radius* DG *of an altitude equal to* BF, *but it shall approach continually to this cylinder, while the figure is produced, as its limit.* FIG. 112.

It is demonstrated, in the same manner as in art. 307. that if B*d* parallel to AP be equal to DG, and F*x* be taken equal to M*m* from F towards B, the cylinder generated by the rectangle *dx* revolving about B*x* shall be always equal to the solid generated by the area *bf*N*n* about the axis AP. Therefore the cylinder generated by the rectangle F*d* about the axis FB is the limit of the solid that is generated by the area *bf*N*n* about AP.

312. In art. 301. let MT touch the curve AM in M, and meet the base in T; and, while the curve AM is produced, the point T shall approach continually to R, the angle PTM to the angle PRS, and consequently the tangent MT shall approach continually in position to the asymptote RS. For, let PY be equal to the given right line DG, and Y*l* parallel to PM meet KI in *l*, and NL parallel to the base in L; and MT the tangent of the curve at M shall be parallel to PL, by art. 247. The right line P*l* is parallel to the asymptote RS, by art. 301. Therefore the angle LP*l* is equal to the angle formed by the tangent MT and asymptote RS: And, since L*l* (or IN) may become less than any given right line by producing the figure while PY remains invariable, so that the angle LP*l* may become less than any given rectilineal angle; it follows, that MT approaches continually to the asymptote RS in position. Hence, when FIG. 111.

the curve is ſuppoſed to be infinitely produced, the tangent at its infinitely diſtant termination is ſaid to coincide with the aſymptote; which therefore is commonly conſidered as the tangent of the curve at an infinite diſtance. In the ſame manner, when two curves are aſymptotes to each other mutually, they are ſaid to touch each other at an infinite diſtance; and, converſely, when both are ſuppoſed to be infinitely produced, if they are ſuch as muſt be ſuppoſed to meet at their termination, they are ſaid to be aſymptotes of each other.

313. Hence the theorems for drawing tangents to curves often lead to ſuch as ſerve for determining their aſymptotes, there being no more required than to find the right line to which the tangent approaches in poſition while the curve is produced continually. Of theſe we ſhall ſubjoin one, that follows from prop. 18. by which the aſymptotes are determined when a curve is ſuppoſed to be deſcribed by the interſection of right lines that revolve about given points, and the ratio of the velocities with which thoſe lines revolve can be found.

PROP. XXVI.

Fig. 117. & 118. *The reſt remaining as in prop.* 18. *let the fluxion of the angle* ACP *be to the fluxion of* aSP (*or the angular velocity of* CP *about* C *to the angular velocity of* SP *about* S) *while* CP *and* SP *become parallel, as* SQ *is to* CQ; *and, if* RX *the aſymptote of the curve deſcribed by* P *meet* CS *in* R, CR *ſhall be equal to* SQ.

It follows from prop. 18. that if PD be the tangent of the curve deſcribed by P, the angle SPT be made equal to CPD according to that propoſition, and PT meet CS in T; the fluxion of the angle ACP ſhall be to the fluxion of the angle aSP (or the angular velocity of CP to the angular velocity of SP) as ST is to CT. Let the angle PCV be always equal to PST, and PD meet SC and CV in D and V; let Cx and Sy be parallel to the aſymptote RX, and, the angle xCv being

being made equal to XRC, let C*v* meet RX in *v*. Because the triangles PCV, PST are similar, CV is to ST as CP is to SP; and the ratio of CV to ST approaches continually to a ratio of equality while P describes the branch of the curve that belongs to the asymptote RX, because while the right lines CP, SP approach to parallelism, their ratio approaches to a ratio of equality: But C*v* is the limit to which CV approaches at the same time, because (by the last article) the tangent PD approaches in position to the asymptote RX, CP to C*x*, and the angle PCV (or PST) to *y*ST, or XRC, or *x*C*v*. The angle C*v*R being equal to CR*v*, C*v* is equal to CR; and therefore CR is equal to the limit to which ST continually approaches while P describes the curve FH. But since the ratio of the fluxion of the angle ACP to the fluxion of *a*SP when CP and SP become parallel, is that of SQ to CQ, by the supposition; it follows, that SQ is the limit to which ST approaches continually while P describes FH. Therefore CR is equal to SQ. If the angles PCS, FIG. 118. PSC increase or decrease together while CP and SP become parallel, the point Q must be taken upon CS betwixt C and S; but it is to be taken upon CS produced beyond C or beyond S, FIG. 117. when one of those angles decreases while the other increases, according as the velocity of SP about S is greater or less than that of CP about C: And it is manifest, that CR and SQ must be always taken in contrary directions from C and S. If one of the angles PCS, PSC decrease and the other increase while the point P describes FH, and the fluxions of those angles be equal when CP and SP become parallel, the point Q is not then assignable, and is said to become infinitely distant; in which case the branch of the curve described by P is not of the hyperbolic kind, unless sometimes when CP coincides with CS.

314. COR. Let SQ be to CQ as the angular velocity of CP is to the angular velocity of SP at the term or moment when those lines become parallel, and let CR be taken equal to SQ with the precautions we have described; then RX drawn parallel to CP or SP (which are supposed parallel to each other) shall be an asymptote of the curve described by P. If the right lines CP and SP be always tangents of any curve lines that pass through C and S, instead of revolving about those

 points,

points, the construction by which the tangent or asymptote of the curve described by P is determined, is the same. The use of the 18th proposition for drawing the tangents of curve lines, and of this proposition for determining their asymptotes, will appear from the following examples, which we chuse from a great number that might be brought.

315. *Ex.* 1. The right lines CA, S*a* being given in position, let the angle ACP be always to *a*SP in any given ratio; determine the point T so that ST may be to CT in the same given ratio, and that T may lie in the same or in an opposite direction from C and S, according as the right lines CP, SP revolve about C and S with the same or with opposite directions; join PT, and make the angle CPN equal to SPT, so that those angles may be on opposite sides of CP and SP: And PN shall be the tangent of the curve described by P. The point T is a given or invariable point, because the ratio of ACP to *a*SP is supposed to be given; let this ratio be that of *m* to *n*, and ST shall be to the given line CS as *m* is to the difference of *m* and *n* when CP and SP revolve about C and S in the same direction, but as *m* is to the sum of *m* and *n* when they revolve in opposite directions. Let CR be taken from C equal to ST in an opposite direction, and the asymptotes of the curve (if it has any, that is, if CP and SP ever become parallel while the curve
FIG. 119. is described) shall pass through R. To mention a few of the simplest cases that are comprehended in this example, let ACP be equal to *a*SP, and if CP and SP revolve about C and S with the same direction, the point P shall describe a circle: In this case PT is to be considered as parallel to CS, and the angle CPN being made equal to SPT or PSC, PN is the tangent of
FIG. 120. the circle, as is well known. But if CP and SP revolve about and S with opposite directions, the point P shall describe an equilateral hyperbola. For if S*a* be a right line given in position, ASP be always equal to SCP, PH parallel to SA meet CS in H; the triangles HPS, HPC being similar, the rectangle SHC shall be equal to the square of PH, and P shall describe an equilateral hyperbola, of which CS is a diameter and SA a tangent at S. Bisect CS in T, join PT, let the angle SPN be made equal to CPT, and PN shall be the tangent of the hyperbola

perbola at P: As in the circle, if SPN be made equal to CP*t*, *t* being the center and CS a diameter, PN is a tangent; only the angle SPN in the circle is taken the contrary way. Some other analogies betwixt the circle and equilateral hyperbola appear alſo from this: 1. Let CS any diameter of the hyperbola be drawn and right lines from C and S to P any point in the hyperbola, if the angle PSA be made equal to PCS, SA ſhall be the tangent at S: The tangent of the circle at S is determined by the ſame conſtruction, only that angle is to be taken on the other ſide of SP. 2. When any two points P and *p* are in the equilateral hyperbola, and right lines are inflected from C and S, the extremities of any diameter CS that paſſes through the center T, to thoſe points, the angle PC*p* is equal to PS*p* or to its ſupplement to two right ones. 3. The angle PTS contained by TP, TS any two ſemidiameters of the hyperbola is equal to the angle PVA contained by SA, PN the tangents at the extremities of thoſe ſemidiameters: For produce PT till it meet the oppoſite hyperbola in K, join CK, CS; and, TV which is parallel to CP (becauſe it biſects CS and SP) and the angle PTV being equal to CPT or SPV, and STV equal to TSK or PSV, it follows, that PTS is equal to PVA. As for the aſymptotes, becauſe the angular velocity of CP is equal to the angular velocity of SP, it follows from the laſt propoſition, that they paſs through T, the point R coinciding with T; and, if PB, P*b* be taken equal to PS, the aſymptotes ſhall be parallel to SB and S*b*, becauſe, when S*p* comes into the poſition of SB or S*b*, then C*p* and S*p* become parallel.

Fig. 119. & 120.

316. If the angle PSC be always double of PCS, take ST from S towards C equal to one third part of SC; and the point P ſhall deſcribe an hyperbola whoſe tranſverſe axis ſhall be CT, and that ſhall have one of its *foci* in S: And, if the angle CPN be made equal to SPT, according to the laſt propoſition, PN ſhall be a tangent. Take CR equal to ST, or one third part of CS, and the aſymptotes ſhall paſs through R conſtituting with RS an angle of 60 degrees, becauſe CP and SP become parallel when the angle PCS is of that magnitude. If the angle PCB be always double of PSB, (the point B being ſituated on CS produced beyond C,) it is obvious, that P ſhall deſcribe Fig. 121.

Fig. 122.

ſcribe a circle paſſing through S having its center in C, and that
FIG. 123. T is the extremity of the diameter through S. If CA and S*a* form any given angles with CS, ACP be always double of *a*SP, and the angular motions of CP and SP be in the ſame direction about the points C and S, the point P ſhall deſcribe a line of the third order that has a double point in S, of the 34th kind according to Sir ISAAC NEWTON's enumeration: And if CT be taken upon SC produced beyond C equal to CS, and the angle SPN be made equal ro CPT according to the laſt propoſition, PN ſhall be the tangent. If CR be taken from C equal to ST but in a contrary direction, the aſymptote ſhall paſs through R; and, S*d* being parallel to CA, if the angle *a*SD be made equal to *a*S*d* on the oppoſite ſide of S*a*, the aſymptote ſhall be parallel to SD. If CP meet the right line SD given in poſition in M, MP ſhall be always equal to MS; and therefore this curve coincides with one, of which ſeveral properties are demonſtra-
FIG. 124. ted elſewhere, *(lemma* 2. *par.* 1. *Deſcrip. curvar.)* If the angular motion of SP about S be triple of the angular motion of CP about C, and be in the ſame direction, ſo that TSP be always triple of TCP, the point P ſhall deſcribe a line of the third order of the ſame *ſpecies* with the former, only in this curve CT is triple of CR. If AM perpendicular to CS biſect it in A, and CP meet AM in M, SP ſhall be equal to SM or CM; becauſe the angles SMP, SPM are equal, being each double of SCP: Therefore, ſince the angle TSP is triple of TCP, the area TSP is triple of the triangle ACM; and if ACK be an angle of 60 degrees, the area bounded by the curve CPT and right line CT ſhall be triple of the triangle CAK. The limit of the area included betwixt CR, the aſymptote RX and the curve, is alſo triple of the ſame triangle. Let C*p*T be the curve called the *foliat*, that has the ſame aſymptote RX, the ſame double point in C, and paſſes through T; let PV perpendicular to RT meet the foliat in *p*, then PV ſhall be to *p*V in the invariable ratio of AK to CA, or of the ſquare root of 3 to unit. And hence the quadrature of the foliat and its tangents may be determined: For the area S*p*T muſt be to to SPT (or 3CAM) as 1 is to $\sqrt{3}$; and, the angle CPN being made equal to SPT, if NP meet RT in *n*, the right lines

P*n*,

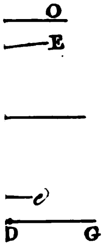


Fig. 115.

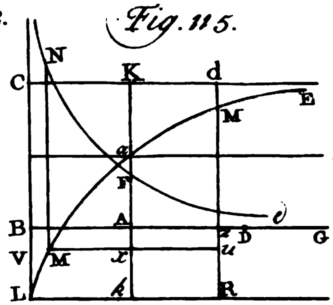
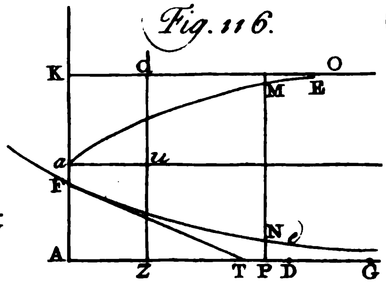


Fig. 116.



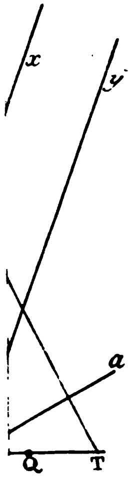


Fig. 118.

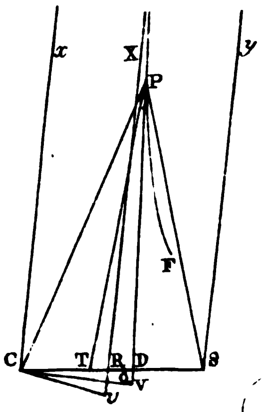


Fig. 121.

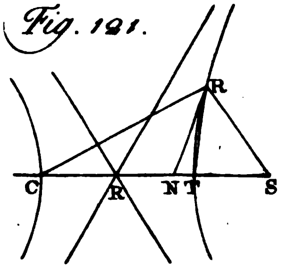


Fig. 122.

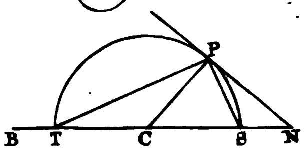
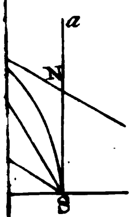
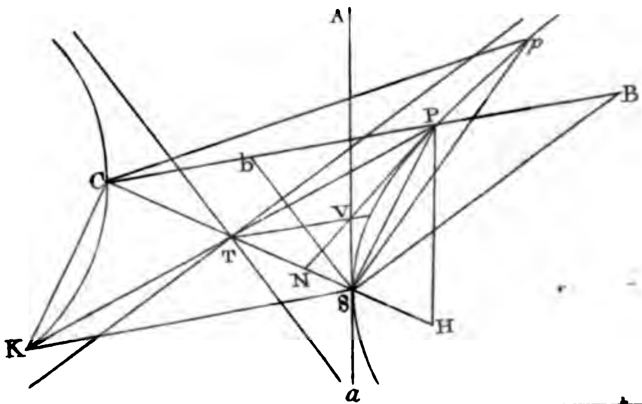


Fig. 120.





Pn, pn shall touch the curves TPC, TpC respectively. It will appear afterwards, that the curve TPC is one of those which can be described by a centripetal force directed towards S that is reciprocally as the cube of SP the distance from S. The equation of this curve is easily computed from hence, that the square of PV is to the square of CV as TV is to RV. If FIG. 125. the angle PSC be triple of PCS, then ST is to be taken from S towards C equal to one fourth part of SC; the curve described by P is also of the third order, having three asymptotes that pass through R, (CR being taken from C towards S equal to one fourth part of CS,) one of which is perpendicular to CS, and the other two form each an angle with CS that is half a right one. In other instances, when the angle ACP is to aSP in a given ratio, the tangents and asymptotes are determined with the same facility.

317. *Ex.* 2. Let the right lines CM, SM be always inflected FIG. 126. from the given points C, S to the curve BM; and, if the point P be taken upon CM so that the triangle SMP may be always isosceles, the tangents and asymptotes of FP the curve described by P may be determined from those of the curve BM by the last proposition. Let EM be the tangent of the curve BM, and first let MP be always equal to MS; make the angle SMV equal to CME on the opposite of SM that CME is of CM; let MV meet CS in V, take ST equal to 2SV, join PT, make the angle CPN equal to SPT the contrary way from CP that SPT is from SP: and PN shall be the tangent of the curve FP. For, the angle MSP being always equal to MPS, it follows, that the angular velocity of SP about S is one half of the sum or of the difference of the angular velocities of CM and SM about C and S, according as ME meets CS produced beyond C or S, or between C and S. Therefore, if we suppose that ST is to CT as the angular velocity of CP is to the angular velocity of SP, one half of the sum of CV and SV, or of their difference, shall be to SV as CT is to ST. Therefore ST must be equal to 2SV; and, when CP and SP become parallel, CR is to be taken equal to 2SV in order to determine the asymptote RX. When M is always in the same right line, P is always in a line of the third order of the 33d sort, and has no double point, unless that right

line

line paſs through S; in which caſe the curve FP coincides with one of thoſe which we conſidered in the laſt article. But if we ſuppoſe that the ſide SP is always equal to SM, then, the reſt of the conſtruction being the ſame, ST is to be taken equal to SV. Laſtly, if PM be always equal to PS, ST is to be taken equal to one half of SV. Of this laſt caſe we have an example in

FIG. 127. the ellipſe and hyperbola: For when M is always found in a circle deſcribed from the center C, P is found in an ellipſe or in an hyperbola according as the point S is within or without the circle, C and S are the two *foci*, CM is equal to the tranſverſe axis, and the angle CME is a right one: Therefore, SMV being made a right angle, ſince SP is equal to PM, if PH biſect SM in H, it ſhall biſect SV in T, and ſhall touch the curve FP; becauſe, the angles SPT, CPN being equal, PN and PT coincide. If, inſtead of a right line revolving about C, we ſubſtitute a right line moving parallel to itſelf, the ſame conſtructions may be eaſily adapted for determining the tangents and aſym-

FIG. 128. ptotes of the curve that ſhall be deſcribed by P: And, ſuppoſing M to be always found in a right line, we ſhall have an example of this in the parabola when MP is always equal to PS, and in the equilateral hyperbola when MP is equal to MS.

FIG. 129. 318. *Ex.* 3. Let the angular velocity of CP about C be to the angular velocity of CM about C as any right line C*m* is to CS; and the angular velocity of SP about S to the angular velocity of SM about S as S*n* is to CS. Then, the reſt of the conſtruction being the ſame as formerly, let ST be to CT in the ratio compounded of that of C*m* to S*n* and that of SV to CV. This and the preceeding examples might be eaſily rendred more general.

FIG. 130. 319. *Ex.* 4. Let the invariable angles DCG, KSH revolve about the given points C and S; let M the interſection of the ſides CD, SK deſcribe the curve BM, and let FP be the curve deſcribed by P the interſection of the other ſides CG, SH. Let AM be the tangent of the curve BM, make the angle SMT equal to CMA, join TP, and make CPN equal to SPT, with the precautions that have been mentioned ſo often, and PN ſhall be the tangent of the curve FP. For the angular velocity of CM about C is to the angular velocity of SM about S as ST is to CT, by prop.

prop. 18. The angular velocities of CP and SP about C and S are respectively equal to the angular velocities of CM and SM about the same points, because the angles MCP, MSP are invariable. Therefore the angular velocity of CP is to the angular velocity of SP as CT is to ST; and, by the converse of the 18th proposition, PN is the tangent of the curve FP. When M comes to *m*, let SP and CP become parallel; and, *ma* being the tangent at *m*, let S*m*Q be made equal to C*ma*; and, if *m*Q meet CS in Q, take CR equal to SQ the contrary way from C that Q is from S, draw RX parallel to CP or SP, and it shall be an asymptote of the curve FP: but if *m*Q be parallel to CS, the branch of the curve described by P while M comes to *m* shall not be of the hyperbolic kind. When the point M describes a right line, P describes a conic section, unless when CP and SP coincide at the same time with CS; in which case P likewise describes a right line. When M describes a conic section that passes through one of the poles C, S only, and the right lines CP, SP coincide not with CS at the same time, P describes a line of the third order that has a double point in that pole. When the conic section passes through neither of the poles, P describes a line of the fourth order that has three double points: And by these constructions the tangents and asymptotes of those curves, and of such as can be reduced to similar classes of the higher orders, are determined. While the invariable angles MCP, MSP revolve about C and S, and M describes a right line, if the given angle P*sp* revolve about another given point *s* and the side *sp* always meet CM in *p*; then shall *p* describe a line of the third order that shall have a double point in C, because P describes a conic section that passes through C. The tangent at *p* is determined by drawing P*t*, so that the angle *s*P*t* be equal to SPT, meeting C*s* in *t*, joining *pt* and constituting the angle C*pn* equal to *spt*: for *pn* shall be the tangent; because the angular velocity of *sp* is equal to that of *s*P, and is to the angular velocity of CP, or of C*p*, as C*t* is to *st*.

320. *Ex.* 5. Let the angle MCP revolve about C as in the preceding example; but, the angle PQM being invariable, let the angular point Q describe the curve IQ, and the side QM always pass through the pole S; let A*a*, B*b* and PN be tan- FIG. 131.

 gents

gents of the curves IQ, GM and FP respectively. Let the angle QSD be made equal to DQa, (so that a circle described through S, Q and D may touch Aa;) join CD, and let the angles DPT, SMH be made equal to the angles CPN, CMB respectively, with the precaution so often mentioned; join TH, and it shall be parallel to SD: And hence, the tangent of IQ being given with the tangent of either GM or FP, the tangent of the other is easily determined. The asymptotes of the branches that are described by M when CM and SM become parallel, or by P when CP and QP become parallel, are determined by a construction that easily follows from this. When the point N describes a right line, and either of the points M, P describe also a right line, the other point describes a line of the third order (some cases excepted) that has a double point. The demonstration of this construction is easily deduced from prop. 18. For it follows from what was shewn there, that the angular velocity of DQ about D is equal to the angular velocity of SQ about S; so that if Aa, PN and Bb are the tangents, CT must be to DT as CH is to SH, and TH parallel to SD.

FIG. 132. 321. *Ex.* 6. The lines of the third order that have a *double* point are also described by P when the right lines CM, DM and SP revolve about the poles C, D and S, so that M describes a conic section that passes through C, and SP is always parallel to DM. Let Mt touch the conic section, and, the angle DMV being made equal to CMt, let MV meet CD in V; draw VH parallel to DS meeting CS in H; and, the angle CPT being made equal to SPH with the usual precaution, then PT shall be the tangent of the curve described by P. One of the asymptotes is parallel to CD, and its position is determined by producing CD till it meet the conic section in E, drawing ~~the tangent of the section~~ EQ parallel to AS meeting CS in Q, and taking CR equal to SQ according to the last proposition. Hence a method is easily derived for drawing a line of the third order through six given points, one of which is to be a double point, ~~that~~ shall have an asymptote parallel to a right line given in position, or a branch whose tangent approaches to coinciding with this right line as its limit, as those curves always have. When M describes any other curve, the tangents and asymptotes of the curve described by P are deter-

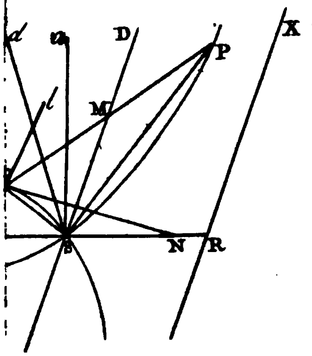


Fig. 124.

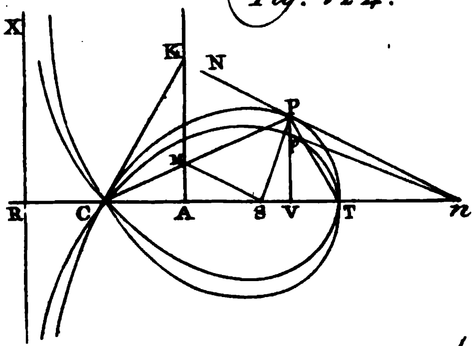


Fig. 126.

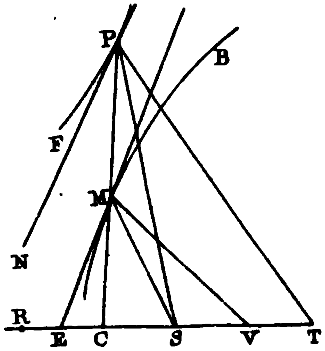
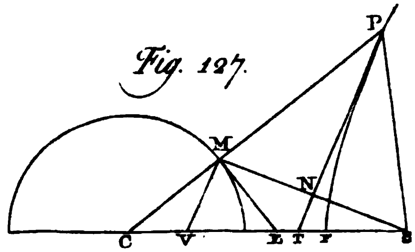


Fig. 127.



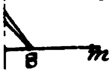
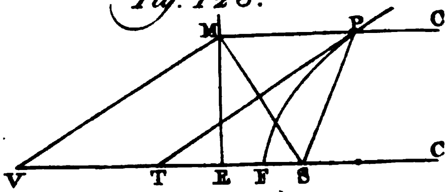


Fig. 128.



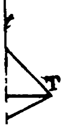


Fig. 130. N. 2

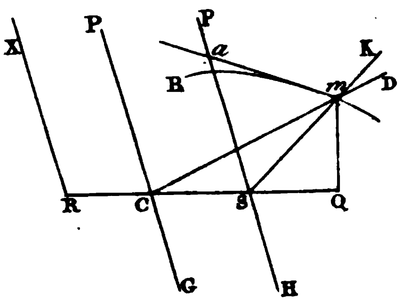
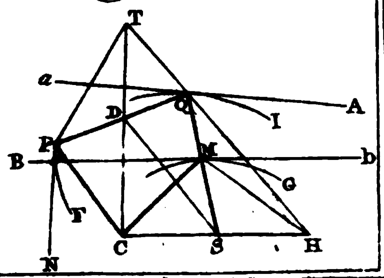


Fig. 131.



determined in the same manner. Thus, in the conchoid, if *m*H parallel to SA meet SH parallel to AD in H, and the angle QMT be made equal to SMH, MT is the tangent. FIG. 108.

322. *Ex.* 7. Let the right lines CM, SQ, DV revolve about the points C, S and D; let the intersection of DV with CM describe the curve GM, and its intersection with SQ describe the curve HQ; let B*b* touch the former curve in M, and A*a* touch the latter in Q. Join DC, DS and CS, make the angle DML equal to CMB, (with the usual precaution,) and let ML meet CD in L; let the angle DQT be equal to SQA, and let QT meet DS in T; let LT meet CS in H, join PH, and make the angle CPN equal to SPH the contrary way from CP that SPH is from SP, and PN shall be the tangent of the curve described by P. The same construction is easily adapted to the case when the right line MQ moves parallel to itself instead of revolving about a given point. The asymptote of the branch that is described when CP and SP become parallel is determined by taking CR equal to SH in that case, according to the last proposition. When M and Q describe right lines, P describes a conic section, if the revolving lines CP and SP do not coincide with CS at the same time, (in which case P describes a right line,) as may be demonstrated by lemma 20. lib. 1. of Sir ISAAC NEWTON's Principles, and by several other methods: And hence a way arises of describing a conic section through five given points similar to the first of those that are given in the 22d proposition of the same book. If more poles be assumed in the same manner, and the intersections of the revolving lines that are necessary in the description for determining P, move in fixed right lines, the point P shall still describe a conic section. But if any of them are made to describe conic sections or curves of any other orders, the line described by P may arise to the order that is expressed by the double product of the numbers that express those orders; and its tangents may be determined from the same principles. In this example, where we make use of three poles only, when the point M describes a conic section that passes through C, or D, and Q describes a right line, the point P describes a line of the third order having a double point in C, or S, (some cases excepted, as when CP and SP coincide together FIG. 133.

with CS:) And hence other methods arise for drawing such curves through seven points, one of which is supposed to be double, and for determining its tangents and asymptotes. The conic section GM may be left out in explaining the manner in which the curve is described, or that in which its tangents and asymptotes are found, by substituting the construction by which the points and tangents of GM are determined, in its place; of which substitution we gave an example at the end of art. 319. This may be easily extended to some classes of the higher orders of lines. The lines that are described in this manner have always one or more of those points that are called *double*, or that are more complex: for though, in particular cases, some of those points become simple; yet one or more still remain double, or the Fig. 134. curve becomes a conic section. Thus, if M describe a conic section that passes neither through C nor D, while Q describes a right line AQ, P describes a line of the fourth order, that has three double points, *viz.* C, S, and the point where DC meets AQ; unless when CP and SP coincide at the same time with CS, in which case the points C and S are no longer double, and the curve described by P is of a lower order; but the curve is then either of the third order and has still a double point, or becomes a conic section: And the lines of the third order, that have no *double* point, cannot be described, when the angles or right lines are all made to revolve about fixed points, or poles, by any method at least published hitherto. When the angular points are carried along right lines, as in the following example, such curves may be comprehended in the description likewise.

Fig. 135. 323. *Ex.* 8. The angles PML, PQL being invariable, let A*a*, B*b* and D*d* be the tangents of the curves described by the points M, Q and L, and let PM always pass through a given point C, and PQ through a given point S; make the angle MCG equal to GM*a* or AML, QSH equal to BQL, (so that a circle through C, M and G may touch A*a*, and a circle through S, Q and H may touch B*b*,) and let CG, SH meet LM and LQ in G and H, respectively; join GH meeting D*d* in I, make the angle GLZ equal to HLI with the precaution we have often mentioned; let CT be to ST as GZ is to HZ, so that the points C, T and S may be in the same situation with respect to each other as G,

I

T and H; and if the angle SPN be made equal to CPT the contrary way from SP that CPT is from CP, then PN ſhall be the tangent of the curve deſcribed by P. When M, Q and L deſcribe right lines, P deſcribes a line of the fourth order, in which C and S are double points, unleſs the revolving lines CP and SP coincide at the ſame time with CS. But in this laſt caſe (that is, ſuppoſing CS to meet A*a* and B*b* in A and B, the angle SBK equal to SQL, and BK to meet D*d* in K, when CML is equal to CAK) P deſcribes a line of the third order, the points C and S are ſimple, (*Deſcript. curvar. prop.* 15. *par.* 1.) and the curve interſects CS in a third point *n*, the ſame to which T comes when M comes to A and Q to B. And thus lines of the third order are deſcribed that have no *double* point, and their tangents and aſymptotes are determined from the deſcription. The ſame principles ſerve for determining the tangents and aſymptotes of the curves that are generated when more lines, angles and poles are made uſe of in the deſcription.

324. Theſe examples may ſerve to ſhew how theorems relating to the deſcription of lines are of uſe in reſolving problems concerning them; and this will farther appear when we come to treat of the curvature of lines. The tangents at the points C and S in the preceeding examples are determined more eaſily from art. 214. in a manner that may be applied in ſome caſes to any other points of the curve. For example, let A, B, C, S and E be five points in a conic ſection, and let it be required to draw a tangent at C; join BC, SE, and let them meet in D; let SC meet AB in *q*, and *q*D meet AE in *n*; join C*n*, and it ſhall be the tangent of the conic ſection at C. Again, let a line of the third order have a double point in C, and let it be required to draw a tangent to any other point in the curve, as S. Let any right line through S meet the curve in other two points *f* and *d*, and let another right line meet the curve in the three points *a*, *b* and *e*; join C*e* meeting S*d* in D. Let a conic ſection deſcribed through the five points C, *a*, *b*, *d* and *f* meet CS in *g*, join D*g* meeting *ab* in *k*, then join S*k*, and it ſhall be the tangent at S. Or the point *g* may be found without deſcribing the conic ſection, by producing *db* and *a*C till they meet in *l*, *l*S and *af* till they meet in *h*, and joining *bh* which produced (if neceſ-

Fig. 136.

Fig. 137.

ſary)

sary) shall meet CS in *g*. This tangent may be determined by several other methods from the same principles, and some properties of such lines may be deduced analogous to those demonstrated of the conic sections by the learned Mr. SIMSON, *Sect. conic. lib.* 5. *prop.* 45. *&c.* Many other examples might be brought of this kind; but we have insisted on this subject at a sufficient length. We have described these examples in this rather than in the seventh chapter, that we might consider the tangents and the asymptotes together.

325. By an assignable magnitude we always understand any finite magnitude, and by an assignable ratio that of any finite quantities.

PROP. XXVII.

FIG. 138. *Let* P*r*, *the ordinate of the figure* AP*rg*, *be to* PN, *the ordinate of the figure* APNF, *in any ratio that approaches to an assignable ratio, as its limit, while the figures are produced; let* PN *be reciprocally as any power of* BP, *and if the exponent of this power be greater than unit, and* AP *be taken upon* BA *produced beyond* A, *the areas* APNF, AP*rg* *shall both have limits; but if that exponent be not greater than unit, those areas may be produced till they exceed any given space. It is the contrary, when* AP *is taken from* A *towards* B.

FIG. 107. Let IM in art. 293. be reciprocally as any power of CI (or BP) whose exponent is any positive number or fraction expressed by *m*, and the fluxion of IM shall be to the fluxion of CI (or of the base AP) in the ratio compounded of that of IM to CI and that of *m* to unit, (art. 167. & 168.) Let PN be to DG (as in art. 293.) as the fluxion of PM (or of IM) is to the fluxion of the base AP: then PM shall be as IM directly and BP inversely, and therefore inversely as some power of BP whose exponent exceeds unit by *m*. When AP is taken upon BA produced beyond A, the area APNF is always less than the rectangle contained

tained by aK and DG, which is its limit, by art. 293. But if AP be taken from A towards B, the area APNF may be produced till it exceed any given space, and has no such limit, by art. 294. Therefore, conversely, if PN be reciprocally as any power of BP whose exponent exceeds unit by any integer number, or by any fraction how small soever, then the area APNF has a limit when AP is taken upon BA produced beyond A, but has no limit when it is taken from B towards A. From this it easily follows, (and it may be likewise shewn from art. 295.) that, when PN is reciprocally as any power of the base whose exponent is less than unit, the area APNF has a limit which it never can exceed if AP is taken from A towards B, but has no such limit if AP is taken upon BA produced beyond A. This being premised, because the base AP is an asymptote of the figure APNF, and the ratio of Pr to PN approaches to an assignable ratio as its limit while the figure is produced, it follows, that AP is also an asymptote of the figure $APrg$. Suppose that Ag is an ordinate at such distance from B, that when AP is taken upon BA produced beyond A, Pr never coincides with any asymptote the figure $APrg$ may have parallel to the ordinates; and since the ratio of Pr to PN is never greater than an assignable ratio while AP is any assignable distance, and the limit to which their ratio approaches (whether by increasing or decreasing) while the figure is produced, is also assignable; it follows, that there may be a ratio of a greater finite quantity to a lesser which may exceed any ratio of Pr to PN. Let Ab be to AF, and the ordinate Px always to PN, in such a ratio; and the figure $APxb$ shall be always to APNF in the same ratio, (by art. 111.) Therefore, if there is a limit which the area APNF never can amount to, there is also a limit which the area $APxb$ never can amount to, which is to the former as Ab is to AF. But Px is always greater than Pr, and the area $APxb$ greater than $APrg$; therefore there is likewise a limit which the area $APrg$ never can amount to. If the area APNF may be produced till it exceed any given space, that is, if it have no limit, then let Px be supposed to be to PN in any invariable ratio less than the least ratio of Pr to PN, and the area $APxb$ shall be always less than $APrg$; but since it is in an invariable ratio

FIG. 138.

FIG. 139.

tio,

tio to the area APNF, it follows that it has no limit. Therefore there is likewise no limit which the area APrg may not exceed. When the figure APNF has an asymptote BV parallel to the ordinates, the figure APrg has the same asymptote: and it appears in the same manner, that the areas bounded by a given ordinate Ag, (which is supposed to be taken so near to B that the figure APrg has no asymptote parallel to the ordinates betwixt A and B,) the asymptote BV and the curves, either have both limits, or may be both produced till they exceed any given space. Therefore what was shewn of the area APNF is applicable to the area APrg.

326. Cor. I. It is obvious that this is agreeable to what is shewn by the celebrated author of the Geometry of infinites, after this manner, in art. 1430. That when the base is infinitely produced, and the ordinate becomes an infinitesimal of any order beneath the first, the area is finite; but when the ordinate is an infinitesimal of the first order, whether it be a compleat infinitesimal (as he expresses it) of that order or not, the area is infinite. For when PN is reciprocally as any power of BP if the exponent of this power exceed unit, PN becomes an infinitesimal of an order beneath the first, according to this doctrine, when BP is supposed infinite; but if that exponent be unit or less than unit, PN is an infinitesimal of the first order.

327. Cor. II. To give an example of this proposition, let Pr be expressed by any fraction whose numerator and denominator consist of terms formed from the powers of BP and given quantities, (as by $\dfrac{Ax^m+Bx^{m-1}+\ \&c.}{ax^n+bx^{n-1}+\ \&c.}$ where x is supposed to represent BP, m and n are any given numbers, A, B, a, b, &c. invariable quantities;) then, if the exponent of the highest power of BP (or x) in the denominator exceed the exponent of its highest power in the numerator, the base shall be an asymptote of the figure APrg; and if this excess be greater than unit, (that is, if n exceed $m+1$,) there shall be a limit which the area bounded by the curve and asymptote never can amount to; or, to make use of the usual expression, it shall have a finite value even when it is supposed to be produced infinitely: But if

if that excess be equal to unit, or less than unit, (that is, if n be equal to $m+1$, or less,) the area APrg shall have no such limit; or, according to the usual expression, it shall become infinite when produced infinitely. If the lowest exponent of BP (or x) in the terms of the denominator of that fraction exceed its lowest exponent in the numerator, the figure shall have an asymptote BV parallel to the ordinates; and when this excess is less than unit, there is a limit which the figure bounded by the curve and the asymptote BV never amounts to; but when this excess is equal to unit, or greater than it, there is no such limit: If any power of Pr be expressed by such a fraction, then the excess we have described is to be divided by the exponent of this power, and the quotient in place of the excess is to be compared with unit. For let PN be reciprocally as the power of BP whose exponent is equal to this quotient, and the ratio of Pr to PN shall approach continually to an assignable limit when the figure is produced: because a term in the value of Pr where BP is of any dimensions, is to a term where it is of any lower dimensions, as some power of BP is to a given quantity; and the latter of these terms is to be neglected in respect of the former, when the limit of the ratio of Pr to PN is required while the figure is produced along the base; but the former in respect of the latter, while it is produced along the asymptote BV. And therefore the limit of the ratio of Pr to PN depends on those terms only of the numerator and denominator of the value of Pr, where the exponents of BP are highest, in the former, and where these exponents are lowest, in the latter case.

328. When the ratio of the fluxion of Pr to the fluxion of PN approaches continually to an assignable ratio, as its limit, while the figures APrg, APNF are produced along the asymptote AP, the ratio of the ordinates Pr and PN to each other approaches likewise to an assignable ratio. For let Ps and Pn the ordinates of the figures APsd, APnf measure the fluxions of Pr and PN respectively, the fluxion of the base being measured by the given line DG. Let AP be produced to any point q, and, Q being any where upon Pq, let the ordinates at Q and q meet the curves FN, gr, fn and ds in M, V, T, Z, m, u, t and z; and let Ps be at so great a distance from A that the figures FIG. 138.

 APnf,

APnf, APfd may have no asymptote parallel to the ordinates beyond Pf. The rectangles contained by Pr and DG, and by PN and DG, are the limits to which the areas Pqzf, Pqtn continually approach while Pq is produced by art. 296. and the ratio of Pr to PN is always the ratio of those limits. The ratio of Pqzf to Pqtn is always less than the greatest ratio of the ordinate QZ to QT, and is greater than the least ratio of QZ to QT, when this ratio is variable; and it coincides with this ratio when it is invariable: Therefore the ratio of Pqzf to Pqtn always approaches to an assignable ratio as its limit while Pq is produced, however great the distance AP may be; and, consequently, the ratio of Pr to PN approaches likewise to such a limit. This is sufficient for our purpose; but it might be shewn, that this limit can be no other than the limit of the ratio of Pf to Pn, or of the fluxion of Pr to the fluxion of PN.

PROP. XXVIII.

329. *The fluxion of the base* AP *being given, let the fluxion of the ordinate* PR *be represented by* Pr, *and the limit of the ratio of* Pr *to an ordinate of an hyperbola that is reciprocally as a power of* BP *whose exponent is n, be assignable : Then the figure* APRa *shall have an asymptote that is parallel to the base, or coincides with it, when n is greater than unit; and there shall be a limit which the area bounded by the curve* aR *and this asymptote never can amount to, if n be greater than* 2.

Fig. 138. & 139.

Let Aa parallel to the ordinates meet the curve in a, and ap parallel to the base meet PR in p. The figure apR has an asymptote parallel to the base when the area APrg has a limit, by art. 298. that is, by the last proposition, when the ratio of Pr (which measures the fluxion of PR) to PN approaches to an assignable ratio as its limit, and PN is reciprocally as a power of BP whose exponent n exceeds unit. This asymptote is determined by taking aK upon Aa from a parallel to pR and in the

the ſame direction, ſo that the rectangle contained by aK and DG may be equal to the limit of the area APrg, and, drawing KI parallel to the baſe, by art. 298. The aſymptote ſometimes coincides with the baſe, as when pR is taken from p towards P and aK is equal to Aa. If the exponent n be not greater than unit, the figure has no aſymptote parallel to the baſe, by art. 298. & 325. In the ſame manner it appears, that the figure has an aſymptote parallel to the ordinates when n is equal to unit or greater; becauſe, AP being taken towards B, the figure APrg may be produced in ſuch caſes till it exceed any given ſpace, by art. 325.

330. To demonſtrate the latter part of the propoſition, let KI be the aſymptote of the figure, that is parallel to the baſe when n exceeds unit, (by the laſt article;) let KI be taken upon CK produced towards O, and let Pn meaſure the fluxion of PN, which is ſuppoſed to be reciprocally as a power of BP whoſe exponent exceeds unit. Then, (by art. 167.) the fluxion of the baſe being given, the fluxion of PN is as PN directly and BP inverſely; and, conſequently, Pn is reciprocally as a power of BP whoſe exponent exceeds 2. Therefore, if the ratio of Pr (that meaſures the fluxion of PR, or of IR) to a quantity that is reciprocally as the ſame power of BP approach to an aſſignable ratio, as its limit, the ratio of Pr to Pn, or of the fluxion of IR to the fluxion of PN, and (by art. 328.) the ratio of IR to PN, ſhall approach to ſuch a limit: And, becauſe there is a limit which always exceeds the area APNF, it follows, (by the 27th propoſition,) that there is likewiſe a limit which always exceeds the area aRIK. But if the exponent n be not greater than 2, the area aRIK may be produced till it exceed any given ſpace, by art. 325. becauſe the ratio of IR to a quantity that is reciprocally as a power of BP whoſe exponent is not greater than unit ſhall approach to an aſſignable ratio, as its limit, in this caſe. It appears in the ſame manner, that if KI be taken towards C and AP towards B, the area aRIK has a limit when n is leſs than 2, but has no limit when n is equal to 2, or greater. To give an example of this propoſition, if the ratio of the fluxion of the ordinate to the fluxion of the baſe be expreſſed by ſuch a fraction as was deſcribed

in art. 327. then, if the excess of the highest exponent of BP (or x) in the denominator above its highest exponent in the numerator be greater than unit, the figure has an asymptote that is parallel to the base, or coincides with it: and if this excess exceed 2, there is a limit which the area bounded by the curve and this asymptote can never amount to; but if the excess be equal to 2, or less than 2, that area may be produced till it exceed any given space.

331. Cor. I. The first part of this proposition, reduced to the stile of the method of infinitesimals, agrees with the rule that was cited in art. 299. That when the figure is supposed to be produced infinitely, and the element of the ordinate becomes an infinitesimal two degrees beneath the element of the base, the figure has an asymptote that is either parallel to the base, or coincides with it. For, when the figure is supposed to be produced infinitely, the ratio of Pr to PN must be supposed to coincide with its limit; and, since PN is reciprocally as a power of BP whose exponent is greater than unit, and the fluxion of the ordinate PR is to the fluxion of the base as Pr is to DG, if these fluxions be represented by infinitely small elements of the ordinate and base, the element of the ordinate must be to the element of the base as a given quantity is to a quantity that is expressed by a power of the infinite line BP whose exponent exceeds unit. This rule is chiefly of use (as the celebrated author observes) when the curve is not geometrical, and when AP and PR do not both enter the quantities that express the relation of their fluxions; for if PR, or any other variable quantity besides the base AP and its powers, enter the value of the fluxion (or of the element) of PR, regard must be had to the ratio of that variable quantity to the base AP when it is supposed infinite: And in applying this rule, the various orders of infinites, and infinitesimals, are best determined by the powers of the base, and their reciprocals. The continuation of this rule, and of that given in art. 326. will appear from what follows.

332. Cor. II. The second part of this proposition may be expressed, according to the method of infinitesimals, thus: The figure apR has an asymptote parallel to the base, and the area bounded by the curve and that asymptote is finite, though infinitely

nitely produced, when, the base being supposed infinite, the element of the ordinate pR becomes an infinitesimal of an order three or more degrees beneath the element of the base, which is always supposed an infinitesimal of the first order; but that area is infinite, when the element of the ordinate is an infinitesimal of an order that is only two degrees beneath that of the element of the base.

333. The fluxion of the base being given, let the ratio of the quantity that measures the second fluxion of the ordinate PZ, to a quantity that is reciprocally as a power of BP whose exponent is n, approach to any assignable ratio as its limit. Then the figure has an asymptote that is either oblique to the base, or parallel to it, or that coincides with it, when n exceeds 2; and there is a limit which the area bounded by the curve and this asymptote never amounts to, when n exceeds 3. This is demonstrated from art. 300. 328. & 330. in the same manner as the last proposition was demonstrated from art. 298. 325 & 328. The asymptote is oblique to the base when the ratio of the first fluxion of the ordinate to the fluxion of the base approaches to an assignable ratio, as its limit; but it is parallel to the base, or coincides with it, when this limit is not assignable. If n be equal to 2, or less, the figure has not such an asymptote; and if n be equal to 3, or any number betwixt 2 and 3, the area bounded by the curve and asymptote may exceed any limit. Fig. 140.

334. In order to comprehend the continuation of those theorems in one view, let us call the asymptote, of the *first order*, when it coincides with the base of the figure; of the *second order*, when it is a right line parallel to the base; of the *third order*, when it is a right line oblique to the base; of the *fourth order*, when it is a common parabola that has its axis perpendicular to the base; and, in general, of the order $r+2$, when it is a parabola the ordinate of which is always as a power of the base whose exponent is r. Then let PZ be the ordinate of the figure, and let its fluxion of any order expressed by m (that is, its first fluxion when m is unit, its second fluxion when m is 2, and so on) be represented by Pr, the fluxion of the base being given. Let PN be reciprocally as any power of BP whose exponent is any number n; and let the ratio of Pr to PN approach

proach continually to any assignable ratio, as its limit, while BP is produced. Then, if the number n be greater than m, the figure shall have an asymptote of the order expressed by $m+1$, or of an order that is expressed by a lesser number. When the excess of n above m is greater than unit, there is a limit which the area bounded by the curve aZ and that asymptote never can amount to, while BP is produced; when that excess is equal to unit, or less, this area may be produced till it exceed any given space: And if n be equal to m, the figure has no such asymptote. Let the figure be now produced on the other side of AF, or AP be taken from A towards B, and when n is equal to m, or greater, the figure has an asymptote parallel to the ordinates, that passes through B; when n is equal to m, or when the excess of n above m is less than unit, there is a limit which the area bounded by the curve and this asymptote never can amount to; when that excess is equal to unit, or greater, this area may be produced till it exceed any given space: But when n is less than m, the figure has not an asymptote through B parallel to the ordinates.

335. The first part of the last article, reduced to the stile of the method of infinitesimals, shews, that when BP is supposed to be produced infinitely, and the *difference* of the ordinate of the order expressed by m becomes an infinitesimal of an order that is as many degrees beneath that of the element (or first *difference*) of the base as there are units in $2m$, then the curve has an asymptote of the order expressed by $m+1$, or of some inferior order; when that *difference* becomes an infinitesimal of an order that is as many degrees beneath that of the element of the base as there are units in $2m+1$, the area bounded by the curve and asymptote is finite though it be produced infinitely; but when the order of that *difference* is only $2m$ degrees beneath that of the element of the base, then the area is infinite: And this shews the continuation of the rules given by the learned author above mentioned. Thus, for example, the base being produced infinitely, when the second *difference* of the ordinate is of an order four or more degrees beneath that of the element of the base, the figure has a rectilineal asymptote that is oblique to the base, or is parallel to it, or coincides with it; but if this *dif-*

difference be of an order that is not ſo many degrees beneath that of the element of the baſe, the curve has not ſuch an aſymptote. If the ſecond *difference* of the ordinate is of an order five or more degrees beneath that of the element of the baſe, the area bounded by the curve and aſymptote is finite though produced infinitely; but if it is not of an order that is five degrees beneath that of the element of the baſe, the area is infinite.

336. It is eaſy to ſee from art. 307. & 309. when the ſolids generated by hyperbolic areas revolving about their aſymptotes have limits, and when they may be produced till they exceed any given ſolid. If *a*ME be any hyperbola, the ſolid generated by APNF has a limit, or not, according as AP is taken upon BA produced beyond A, or from A towards B, by art. 301. and ſince IM the ordinate of the hyperbola *a*ME to its aſymptote KI is reciprocally as ſome power of BP, its fluxion is reciprocally as a power of BP whoſe exponent exceeds unit, and, the ſquare of PN being always as this fluxion, (art. 307) PN muſt be as ſome power of BP whoſe exponent is greater than $\frac{1}{2}$. Therefore, when PN is reciprocally as a power of BP whoſe exponent is greater than $\frac{1}{2}$, the ſolid generated by APNF about AP has a limit, or not, according as AP is taken upon BA produced beyond A, or from A towards B. When *a*ME is a logarithmic that has BV for its aſymptote, the fluxion of PM is reciprocally as BP, and PN is reciprocally as the power of BP whoſe exponent is $\frac{1}{2}$; and the ſolid generated by APNF about AP has no limit in either caſe, becauſe the cylinder generated by the rectangle *ag* may exceed any given ſolid. When *a*ME is a parabola that touches BV in L, the fluxion of PM is reciprocally as a power of BP whoſe exponent is leſs than unit, and PN is reciprocally as a power of BP whoſe exponent is leſs than $\frac{1}{2}$: And in this caſe, when AP is taken from A towards B, the ſolid generated by APNF has a limit which it can never amount to, *viz.* the cylinder generated by the rectangle *a*R about the axis *ak*; but it may exceed any given ſolid when AP is taken on the other ſide of A. When PN is reciprocally as a power of BP whoſe exponent is any fraction greater than $\frac{1}{2}$, but not greater than unit, and AP is taken upon

Fig. 114.

Fig. 114. n. 2.

Fig. 115.

Fig. 114.

on BA produced from A, there is no limit which the area APNF may not exceed: but there is a limit which the solid generated by this area never can amount to; and therefore, in this case, when the figure is supposed to be produced infinitely, an infinite area is said to generate a finite solid. If that exponent be still greater than $\frac{1}{2}$, but less than unit, and AP be taken towards B, there is a limit which always exceeds the area APNF, but there is none which always exceeds the solid generated by it; and, in this case, when the figure is supposed to be produced infinitely, a finite area is said to generate an infinite solid. In the former case, the fluxions of the solid and area both decrease while AP is produced; but the fluxion of the solid decreases faster, and is measured by a figure which is to the solid contained by the rectangle P*b*, that measures the fluxion of the area, and by the given line AF, in a ratio that, by producing AP, becomes less than any given ratio; and hence it is not surprising, that the solid should have a limit, though the area has none. In the latter case, that ratio by producing the figures becomes greater than any given ratio; and it is therefore easy to conceive, that though the area has a limit, the solid may have none. FIG. 138. What has been shewn of the solid generated by the area APNF, is to be extended to the solid generated by the area AP*rg* when the ratio of P*r* to PN (however variable it may be) approaches continually to an assignable ratio, as its limit, while the figures are produced along their asymptotes. And, in general, what was shewn of areas, may be transferred, with some necessary precautions, to solids. For example, the fluxion of the base being given, if the ratio of the fluxion of PR to a quantity that is reciprocally as a power of BP whose exponent is n, approach to an assignable ratio, as its limit, and n exceed 2, there is a limit which always exceeds the solid generated by the area *a*RIK about the axis AP, which is supposed to be taken upon BA produced from A.

337. There are several other methods by which it may be discovered when a figure has an asymptote, and of what kind it is. When the value of the ordinate is resolved into a series that converges the faster the greater the base is, the asymptote may be determined from the first terms of the series when they are such

ſuch as remain invariable, or continually increaſe, while the baſe is produced. If the difference betwixt the ſubtangent and the abſciſs approach continually to a finite right line, as its limit, while the branch of the curve is produced, and the baſe (or the ordinate) may increaſe till it exceed any given right line, it is obvious that the figure muſt have a right line for its aſymptote, that meets the baſe at a diſtance from the beginning of the abſciſs equal to that limit, by art. 313. When the curve has a common parabola for its aſymptote, the ratio of the ſubtangent to the abſciſs approaches continually to that of 2 to 1, when the axis of the parabola coincides with the baſe; but to that of 1 to 2, when the axis is perpendicular to the baſe: and by obſerving the limit to which the ratio of the ſubtangent and abſciſs approaches, parabolic aſymptotes, of various kinds, may be diſcovered. When the logarithmic NF is continued on the other ſide of AF, this ratio decreaſes and may become leſs than any given ratio, and no parabolic figure can be its aſymptote. FIG. 116.

338. If SP perpendicular from a given point S upon LP the tangent of the curve LB approach continually in poſition and magnitude to any aſſignable right line, and SL may increaſe without end, it is obvious that the curve BL muſt have a rectilineal aſymptote. If ST, which is perpendicular on PT the tangent of the curve DP, (and is a third proportional to SL and SP,) approach in the ſame manner to a finite right line, the curve DP has a right line, and BL a common parabola, for its aſymptote. If SK perpendicular on TK approach thus to an aſſignable limit, the curve GT has a right line, DP a common parabola, and BL a ſemicubical parabola, for their aſymptotes. In general, when SL may be increaſed without end, if the ratio of SL to a right line that is as any power of SP whoſe exponent is greater than unit, approach to an aſſignable ratio, as its limit, a parabola, the ordinate of which is always as the ſame power of the abſciſs, may be the aſymptote of the curve BL; but if the ratio of SL to SP approach to ſuch a limit, the figure has no rectilineal or parabolic aſymptote. In the ſame manner, if the ratio of the fluxion of the angle ASP to the fluxion of the angle ASL approach to the ratio of any number m to a greater number n, while BL is produced, a parabola whoſe ordinate is al- FIG. 105.

ways as the power of the abscisſ of the exponent $\frac{m}{n}$ is the aſymptote of the curve BL. There are ſtill other rules by which it may be ſometimes diſcovered when a figure has an aſymptote and the area a limit, but we have inſiſted on this ſubject already at a ſufficient length.

PROP. XXIX.

339. *The figure* APNF *being ſuppoſed to revolve about its aſymptote* AP, *there is a limit which always exceeds the ſurface generated by the curve* FN, *when there is a limit which always exceeds the area* APNF; *but when this area may be produced till it exceed any given ſpace, the ſurface generated by* FN *has no limit.*

FIG. 141.

Let N*t* be the tangent at N, and let *xt* parallel to PN meet N*t* in *t*, and meet N*x* parallel to the baſe and equal to DG (which meaſures the fluxion of the baſe) in *x*. The fluxion of the ſurface generated by FN is meaſured by a ſpace that is to the rectangle contained by PN and N*t* in the invariable ratio of the circumference of a circle to its radius; this rectangle is always greater than the rectangle P*x* which meaſures the fluxion of the area APNF, becauſe N*t* is always greater than N*x*: Therefore the ſurface generated by FN is always greater than the area APNF; and, conſequently, if this area may be produced till it exceed any given ſpace, the ſurface FN may likewiſe be produced till it exceed any given ſpace: which is one part of the propoſition. Let FT the tangent at F meet the aſymptote in T, and A*g* be to AF as FT is to AT; let P*r* be to PN in the invariable ratio of A*g* to AF, and *rz* parallel to the baſe meet *tx* produced in *z*. Becauſe the curve is convex towards its aſymptote, *tx* continually decreaſes while the figure is produced, (art. 184.) and the ratio of N*t* to N*x* decreaſes, and is always leſs than the ratio of FT to AT or of P*r* to PN; and the rectangle contained by PN and N*t* is leſs than the rectangle

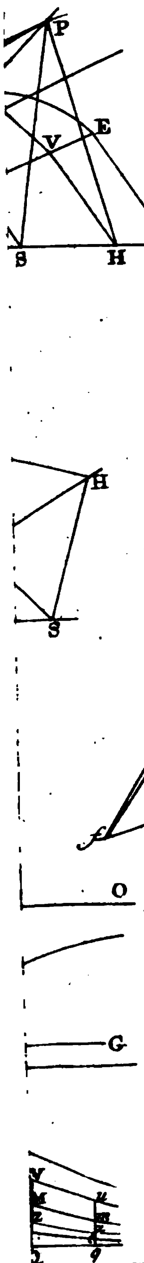


Fig. 133.

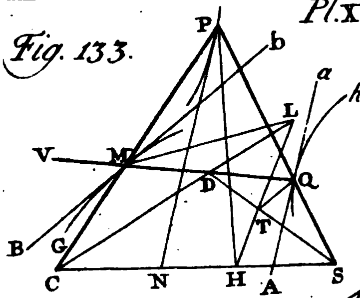


Fig. 134.

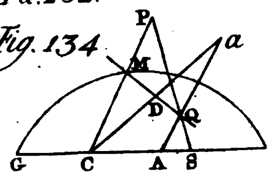


Fig. 137.

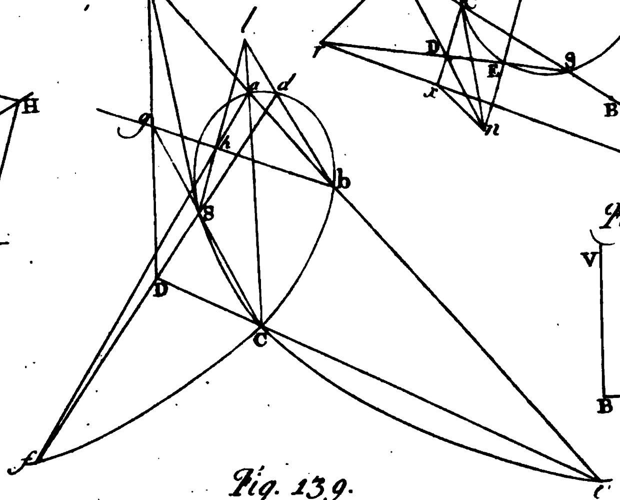


Fig. 136.

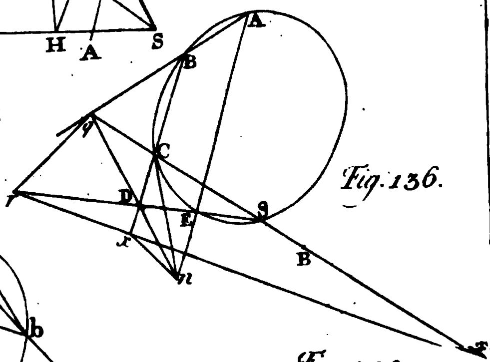


Fig. 140.

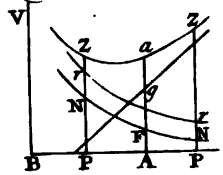


Fig. 139.

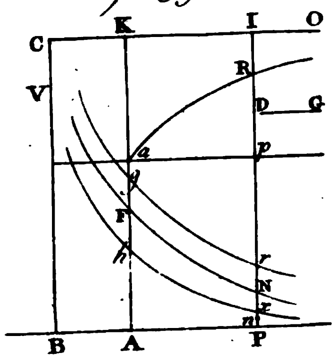
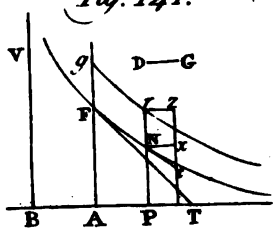


Fig. 141.



angle Pz which measures the fluxion of the area $APrg$. Therefore the fluxion of the surface generated by the curve FN about AP is always to the fluxion of the area $APrg$, in a less ratio than the circumference of a circle is to its radius; and the ratio of that surface to the area $APrg$ is always less than this ratio. But if there is a limit which always exceeds the area APNF, there is likewise a limit which always exceeds the area $APrg$, because the ordinates PN, Pr are supposed to be to each other in an invariable ratio; and therefore there is also a limit which always exceeds the surface generated by the curve FN about AP. Thus it appears, to express the proposition in the more usual form, that when the figure is supposed to be produced infinitely, and to revolve about its asymptote, the surface generated by the curve is finite or infinite according as the area of the figure is finite or infinite; as has been observed by Mr. Cotes. If the figure, for example, is that of the logarithmic, and AP be the asymptote; or if it is an hyperbola, and the ordinate PN be reciprocally as any power of BP whose exponent exceeds unit: there is always a limit which exceeds the surface generated by the curve about the asymptote AP, when AP is taken upon BA produced beyond A: But when AP is taken from A towards B, the surface generated by FN may exceed any given space, as might be demonstrated from prop. 27.

340. As a curve may approach continually to a right line while they are both produced, and never meet it, so a spiral line may approach continually to a certain point, and not reach it in any number of revolutions how great soever that can be assigned. Let a circle be described from the center S with a radius Sb equal to AF in fig. 114. let b be a point given in the circumference of this circle; let the arch br be always equal to the base AP, and Sm be taken on the ray Sr always equal to the ordinate PN; and, since AP is supposed to be the asymptote of the figure APNF, it is manifest that the spiral br approaches continually to the point S, as the curve approaches to its asymptote; but it can never reach it in any assignable number of revolutions, because though the arch br (equal to AP) be never so great, Sm (equal to PN) never vanishes. Fig. 142. & 114.

PROP. XXX.

FIG. 142. & 114. 341. *The construction in art.* 307. *being supposed, let the arch hr be always equal to* AP, *and* S*m equal to* PN; *and the spiral area* S*hm shall be always to the rectangle ag in the invariable ratio of* DG *to* H*h the diameter of the circle hr*H.

By what was supposed in art. 307. the square of PN is to the square of DG as the fluxion of PM is to the fluxion of AP, or (if M*l* parallel to the base be equal to DG, and *lt* parallel to the ordinates meet the tangent M*t* in *t*) as *lt* is to M*l*, or DG; and therefore the square of PN is equal to the rectangle contained by DG and *lt*. The fluxion of the spiral area S*hm* is to the fluxion of the sector S*hr* (by prop. 5.) as the square of S*m* is to the square of S*h*. The fluxion of the base AP, or of the arch *hr*, being measured by DG, the fluxion of the sector S*hr* is measured by one half of the rectangle contained by S*h* and DG, and is to the fluxion of the rectangle *ag* (which is measured by the rectangle contained by *au*, or DG, and *lt*) as S*h* is to 2*lt*. Therefore the fluxion of the spiral area S*hm* is to the fluxion of the rectangle *ag* as the square of S*m*, (or of PN,) or the rectangle contained by DG and *lt*, is to the rectangle contained by S*h* and 2*lt*; that is, as DG is to 2S*h* or 2AF: and, this ratio being invariable, it follows, that the spiral area S*hm* is to the rectangle *ag* as DG is to H*h* the diameter of the circle *hr*H. This proposition obtains, whether AP and KI be asymptotes of the curves FN, *a*M or not; it is sufficient that the square of PN be to the square of DG as the fluxion of PM is to the fluxion of AP, that *hr* be always equal to AP, S*m* equal to PN, and that PM either increase or decrease continually.

342. COR. I. Because the sector S*hr* is equal to one half of the rectangle FP, it follows, that the spiral area S*hm* is to the sector S*hr* as the solid contained by *ag* and DG to the solid contained by FP and S*h*, or AF, and therefore as the cylinder generated by *ag* about *ax*, or the solid generated by APNF about AP,

AP, to the cylinder generated by FP about AP. Thus, when FN is a right line, *br* is the spiral of Archimedes, the area S*bm* is to the sector S*bq* as the frustum of a cone generated by the trapezium APNF about the axis AP is to the cylinder generated by the rectangle FP about the same axis; and if S*y* be the tangent of the spiral at S, the spiral area S*wmb* shall be to the sector S*by* as one is to three; as was shewn after ARCHIMEDES in the introduction. FIG. 142. n. 2.

343. COR. II. When there is a limit which always exceeds the solid generated by the area APNF about AP, there is likewise a limit which always exceeds the spiral area S*bm* generated by the ray S*m* while it revolves about S; and this limit is to the rectangle *ad* as DG is to H*b*. But if the solid generated by the area APNF produced may exceed any given solid, the area that is generated by the ray S*m*, while it revolves continually about S, may likewise exceed any given space. In the same manner, S*b* being equal to *a*K, if *br* be always equal to *ap*, and S*m* to *p*M, the area generated in this case by *rm* shall have a limit, or not, according as the solid generated by the area *a*MIK about the axis *ap* has a limit, or may be produced till it exceed any given solid: And in this case the circle *br*H is the asymptote of the spiral. If S*m* be always equal to PZ in fig. 140. the spiral of Archimedes is the asymptote of the spiral *bm*: and this might be carried further by art. 334. &c.

344. COR. III. Let FN*e* be a common hyperbola, BA and BC its asymptotes, and *bm* shall be the curve that is called the *reciprocal* or *hyperbolic spiral*; let the arch *hb* be equal to BA, join S*b*, let a circle described from the center S through *m* meet S*b* in *o*, and the arch *mo* shall be of an invariable magnitude equal to *hb*, or BA; for *br* is to *mo* as S*b* is to S*m*, or AF to PN, and therefore as BP (or *br*) to BA. The spiral area S*mb* is to the rectangle *ag* as DG is to 2AF, and therefore in this case S*mb* is equal to one half of the rectangle contained by BA and the difference of AF and PN, or by *hb* and *mr*; and the limit of this area is equal to the sector S*hb*. Let ST perpendicular to S*m* meet the tangent of the spiral in T, and ST shall be of an invariable magnitude equal to *hb*, or BA; because S*m* is to ST in the ratio compounded of that of the fluxion of S*m* FIG. 143. & 114.

to

to the fluxion of *br* (or AP) and of the ratio of S*b* (or AF) to S*m*; that is, in the ratio of AF to BP, or of PN (or S*m*) to BA. It is obvious, that a right line *al* parallel to S*b*, at a diſtance S*a* from it equal to BA on the ſame ſide that *b* is from *b*, is an aſymptote of this ſpiral continued without the circle *bb*.

345. Cor. IV. Let Sp perpendicular from S on *m*T the tangent of the hyperbolic ſpiral (deſcribed in the laſt corollary) meet it in p, and the point p ſhall be always found in the ſpiral whoſe rays decreaſe *proportionally* while the length of the curve *bp* increaſes uniformly, which is called by Mr. Cotes the *complicated tractrix*. For if the angle Sp*t* be made equal to S*m*T, p*t* ſhall be the tangent of the curve in which p is always found, by art. 211. and if S*t* perpendicular to S*p* meet *pt* in *t*, the triangles pS*t*, SpT ſhall be ſimilar and equal, and p*t* the tangent of this ſpiral is always equal to the invariable line ST. The fluxion of this ſpiral *a*p is to the fluxion of the ray Sp as the invariable tangent *pt* (or ST) is to the ray Sp, and therefore Sp decreaſes *proportionally* while *a*p increaſes uniformly. Hence the points of this ſpiral may be found by taking S*m*, upon any ray S*r*, in the ſame ratio to S*r* as the given arch *bb* is to the arch *br*, conſtituting ST perpendicular on S*m* equal to the invariable line ST, or to the arch *bb*, joining *m*T and drawing
Fig. 144. Sp perpendicular to *m*T in p. Or, let a circle *kuf* be deſcribed with the radius *ſk* equal to ST, and let *ſz* meet the tangent *kz* in *z* and the circle in *u*; let *ux* be perpendicular on S*k* in *x*; and, if the arch *ky* be equal to the exceſs of the tangent *kz* above the arch *ku*, and S*n* be taken upon S*y* always equal to S*x*, (the coſine of *ku*,) then *n* ſhall be a point in this ſpiral. A conſtruction of this ſpiral is given by Mr. Cotes, *Harmon. menſur. p.* 84. It is likewiſe conſidered by Mr. Varignon, *Mem. de*
Fig. 143. *l'acad. roy.* 1704. All thoſe ſpirals in which the ratio of the fluxion of the curve to the fluxion of the ray from S, is the ſame as the ratio of ſome power of the given line S*b* to the ſame power of that ray, are conſtructed from this ſpiral by taking the angle *b*S*q* to *b*Sp as unit is to the exponent of that power, and S*q* ſo that the ratio of the ſame powers of S*q* and S*b* may be that of Sp to S*b*; for *q* ſhall be found in ſuch a ſpiral.

PROP.

PROP. XXXI.

346. *When there is a limit which always exceeds the area* APNF *while it is produced along the aſymptote* AD, *there is likewiſe a limit which always exceeds the length of the ſpiral line bm while it approaches to* S; *but if the area* APNF *produced may exceed any given ſpace, the ſpiral line bm may be continued till it exceed any given line.* FIG. 145.

Let *m*T touch the ſpiral in *m*, and meet ST perpendicular to S*m* in T. The fluxion of the baſe AP, or of the arch *br*, being repreſented by any given right line DG, let P*n* the ordinate of the figure AP*nf* meaſure the fluxion of PN or S*m*; let P*r*, the ordinate of the figure AP*rg*, be always to PN in the invariable ratio of DG to SH; let the ſquare of P*q*, the ordinate of the figure AP*qd*, be always equal to the ſum of the ſquares of P*r* and P*n*; and, *qx* parallel to the baſe being made equal to DG, complete the rectangle P*x*. Let a circle deſcribed from the center S through *m* meet S*b* in *o*; and, while *m* proceeds in the ſpiral, the fluxion of *om* ſhall be to the fluxion of *br* (or of AP,) which is meaſured by DG, as S*m* (or PN) is to S*b*; but P*r* is to DG as PN is to S*b*: Therefore the fluxion of *om* is repreſented by P*r*; and, ſince it is to the fluxion of S*m* as ST is to S*m* (by prop. 16.) it follows, that ST is to S*m* as P*r* is to P*n*, and *m*T to S*m* as P*q* is to P*n*. Therefore the fluxion of the ſpiral line *bm* is to the fluxion of S*m* as P*q* is to P*n*, and to the fluxion of AP as P*q* is to DG; and the rectangle contained by DG and the right line that meaſures the fluxion of *bm* is always equal to the rectangle P*x* that meaſures the fluxion of the area AP*qd*. Therefore the rectangle contained by the ſpiral line *bm* and the given right line DG is always equal to the area AP*qd*. But P*q* is always leſs than the ſum of P*n* and P*r*, becauſe the ſquare of P*q* is leſs than the ſquare of the ſum of P*n* and P*r*, by the ſuppoſition; and, conſequently, the area AP*qd* is always leſs than the ſum of the areas AP*rg*, AP*nf*. The rectangle con-

tained

tained by AF and DG is always greater than the area AP*nf*, by art. 296. Therefore the rectangle contained by the spiral line *bm* and DG is always less than the sum of the area AP*rg* and the rectangle contained by AF, DG. Because P*r* is to PN in the invariable ratio of DG to SH, the area AP*rg* is to the area APNF in the same ratio; and if there is a limit which always exceeds the area APNF, there must likewise be a limit which always exceeds the area AP*rg*; and there is consequently a limit which exceeds the rectangle contained by the spiral *bm* and DG. Therefore there is a right line which always exceeds the spiral *bm*. If, by producing the figure, the area APNF may exceed any given space, the area AP*rg*, and the rectangle contained by the spiral *bm* and DG, (which is always greater than AP*rg*,) may likewise exceed any space, and the spiral *bm* may exceed any given line.

347. Cor. I. This theorem, expressed according to the more usual form, is, That when the figure APNF is produced infinitely, and the spiral *bm*, after having made an infinite number of revolutions, reaches the point S, the length of the spiral is finite or infinite, according as the area APNF is finite or infinite. When it is expressed in this manner, it is one of the paradoxes of this kind that has the most mysterious appearance; but there is nothing more wonderful in it, in the manner it is here proposed, than that a line may be continually increasing, and the increments acquired by it decrease in such a manner that it shall never amount to a given line. See art. 289.

348. Cor. II. If the ray S*m* be reciprocally as any power of the arch *br* whose exponent exceeds unit, or if the ratio of S*m* to a quantity that is reciprocally as such a power of *br* approach to an assignable ratio, as its limit, while the figure is produced, there is a certain limit which always exceeds the length of the spiral *bm*. But if S*m* be reciprocally as a power of the arch *br* whose exponent is equal to unit, or less, the spiral *bm* may be continued till it exceed any given line.

349. Cor. III. When FN*e* is the logarithmic curve, S*m* decreases *proportionally* while *br* increases uniformly, and the fluxion of S*m* (or PN) is to the fluxion of *br* (or AP) as S*m* is to an invariable line S*c* that is equal to the subtangent of the logarithmic.

garithmic. The fluxion of *br* is to the fluxion of *om* as S*b* is to S*m*; and therefore the fluxion of S*m* is to the fluxion of *om* as S*b* is to S*k*; and, S*m* being to ST in the same ratio, the angle S*m*T is invariable. Therefore the fluxion of the spiral *bm* is to the fluxion of *rm* in the invariable ratio of *m*T to S*m*; and if *rt* perpendicular to S*r* meet T*m* produced in *t*, the spiral line *bm* shall be equal to the right line *mt*; and *m*T is the limit to which the spiral line produced from *m* continually approaches. This curve is called the *logarithmic* spiral.

350. As a right line, or figure, may increase continually and never amount to a given line, or area; so there are progressions of fractions which may be continued at pleasure, and yet the sum of the terms be always less than a certain finite number. If the difference betwixt their sum and this number decrease in such a manner, that by continuing the progression it may become less than any fraction how small soever that can be assigned, this number is the *limit of the sum of the progression*, and is what is understood by the value of the progression when it is supposed to be continued infinitely. These limits are analogous to the limits of figures which we have been considering, and they serve to illustrate each other mutually. The areas of figures cannot be expressed in many cases but by such progressions; and when the limits of figures are known, they may be sometimes applied with advantage for approximating to the sums of certain progressions. Let the terms of any progression be represented by the perpendiculars AF, BE, CK, HL, &c. that stand upon the base AD at equal distances; and let PN be any ordinate of the curve FN*e* that passes through the extremities of those perpendiculars. Suppose AP to be produced; and according as the area APNF has a limit which it never amounts to, or may be produced till it exceed any given space, there is a limit which the sum of the progression never amounts to, or it may be continued till its sum exceed any given number. For let the rectangles FB, EC, KH, LI, &c. be completed, and, the area APNF being continued over the same base, it is always less than the sum of all those rectangles, but greater than the sum of all the rectangles after the first. Therefore the area APNF and the sum of those rectangles either both have limits, or both have none; and it is FIG. 146.

 obvious,

obvious, that the same is to be said of the sum of the ordinates AF, BE, CK, HL, &c. and of the sum of the terms of the progression that are represented by them. If the curve FN*e*, for example, be the common hyperbola, *b* the center, *b*P the asymptote; and, AB being equal to *b*A, if AF represent unit, the series of ordinates shall represent the progression 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. which therefore may be continued till it exceed any given number, as the hyperbolic area may be produced till it exceed any given space, by art. 297. But if FN*e* be an hyperbola of any higher order, so that the ordinate PN be reciprocally as any power of the base *b*P whose exponent is greater than unit, the area APNF, and the sum of the progression that is represented by the series of ordinates, shall have limits. Thus there is always a limit of the sum of the fractions that have unit for their common numerator, and the squares, cubes, or any other powers of thenumbers 1, 2, 3, 4, &c. whose exponents exceed unit, for their successive denominators.

351. And here we may apply what was shewn above, in art. 325. &c. concerning areas, to progressions. If the ratio of the last term of the progression to a fraction that is reciprocally as any power of the number represented by *b*P, approach to an assignable ratio, as its limit; then there is a limit of the sum of the progression, or not, (or, as it is usually expressed, the value of the progression continued infinitely is finite, or infinite,) according as the exponent of that power exceeds unit, or not. If the terms of the progression are such as arise by substituting successively any numbers that increase by equal differences in place of x in the quantity $\dfrac{Ax^m + Bx^{m-1} + \&c.}{ax^n + bx^{n-1} + \&c.}$ described in art. 327. the progression shall always have a limit when the excess of n above m is greater than unit, but it may be continued till the sum of the terms exceed any given number when this excess is equal to unit, or less. Thus, if the figurate numbers * of any order

* The figurate numbers of the first order are 1, 1, 1, 1, 1, &c. These of the second order are the successive sums of those of the first order, *viz*. 1, 2, 3, 4, 5, &c. and form an arithmetical progression. These of the third order are the

order be divided successively by the corresponding figurate numbers of any order that is two or more degrees higher, the sum of the fractions that shall arise in this manner shall have a limit; but if they are divided by the figurate numbers of the next superior order, the progression of the quotients may be continued till it exceed any given number. In like manner it follows from what was shewn in art. 329. that when the terms of a progression increase, but their successive differences decrease, and the ratio of the last difference to a number that is always reciprocally as any power of bP approaches to an assignable ratio, as its limit; then there is a limit which always exceeds the terms of this progression, or not, according as the exponent of that power is greater than unit, or not. If this exponent be greater than 2, and the terms be subducted successively from their limit, a new progression shall be formed the sum of which shall have a limit.

352. When the area APNF has a limit, we not only conclude from this, that the sum of the progression represented by the ordinates has a limit; but when the former limit is known,

the successive sums of those of the second, *viz.* 1, 3, 6, 10, 15, *&c.* and are the triangular numbers. These of the fourth order are the successive sums of those of the third, *viz.* 1, 4, 10, 20, 35, *&c.* and are the pyramidal numbers, and so on. The figurate numbers of any order may likewise be found, without computing those of the preceeding orders, by taking the successive products of as many of the numbers 1, 2, 3, 4, 5, *&c.* in their natural order, as there are units in the number which denominates the order of figurates required, and dividing always those products by the first product. Thus the triangular numbers are found by dividing the products 1×2, 2×3, 3×4, 4×5, 5×6, *&c.* each by the first product 1×2. The pyramidals are found by dividing the products $1\times2\times3$, $2\times3\times4$, $3\times4\times5$, $4\times5\times6$, *&c.* each by $1\times2\times3$. In general, the figurate numbers of any order denoted by m are found by substituting successively 1, 2, 3, 4, 5, *&c.* in place of x in the general expression $\frac{x \cdot \overline{x+1} \cdot \overline{x+2} \cdot \overline{x+3} \cdot \&c.}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \&c.}$ where the factors in the numerator and denominator are supposed to be multiplied by each other, and to be continued till the number in each be equal to that which expresses the order of the figurates required diminished by unit. And when a figurate number of any order is divided by the corresponding figurate of any higher order, the numerator of the quotient is invariable, and x is in its denominator of as many dimensions as there are units in the difference of the numbers that denote those orders.

we may by it approximate to the value of the latter. Let AB represent unit; then the sum of the rectangles FB, EC, KH, LI, *&c.* and the sum of the ordinates AF, BE, CK, HL, *&c.* shall be expressed by the same number But because the curve is convex towards the asymptote, the sum of those rectangles always exceeds the curvilineal area over the same base, by the triangles FEQ, EKS, KLT, LMZ, *&c.* and by the curvilineal spaces F*u*E, E*x*K, K*z*L, L*y*M, *&c.* The sum of those triangles approaches to one half of the rectangle FB, as its limit; therefore the sum of the rectangles FB, EC, KH, LI, *&c.* is greater than the curvilineal area AFEKLMI that is over the same base added to one half of the rectangle FB, the excess being the sum of the spaces F*u*E, E*x*K, K*z*L, L*y*M, *&c.* that are bounded by the arches F*u*E, E*x*K, K*z*L, *&c.* and their chords FE, EK, KL, *&c.* Hence, when AF is a term at a great distance from the beginning of a progression of this kind, the number that expresses the limit of the curvilineal area APNF added to one half of the term AF gives nearly the value of AF and the subsequent terms BE, CK, *&c.* the spaces F*u*E, E*x*K, *&c.* being neglected. But we may approximate to the value of such a progression more accurately in the following manner, that is deduced from art. 255. & 296. and will appear fully afterwards. Let the limit of the area APNF be expressed by A, the ordinate AF by a, the first fluxion of AF (the fluxion of the base being measured by AB, or unit) by b, the third, fifth and the subsequent fluxions of AF taken alternately, and always positively, by d, f, *&c.* Then the sum of the progression represented by AF, BE, CK, HL, *&c.* shall be found nearly by computing $A + \frac{1}{2}a + \frac{1}{12}b - \frac{1}{720}d + \frac{1}{30240}f +$ *&c.* Or, if AR be taken towards b equal to one half of AB, the ordinate at R meet the curve in V, the limit of the area RPNV (or its value when RP is supposed to be produced infinitely) be now expressed by A, the first, third, fifth and subsequent fluxions of RV taken alternately, and always positively, by b, d, f, *&c.* then the sum of the terms AF, BE, CK, HL, *&c.* may be found by computing $A - \frac{1}{24}b + \frac{7}{5760}d - \frac{31}{967680}f +$ *&c.* When it is not the limit of the progression that is required, but the sum of any number of terms of which AF is the first and af the last; then we may approximate to this sum by the first series if we sup-

suppose A to represent now the curvilineal area AafF, a the difference of AF and af, b the difference of their first fluxions, d the difference of their third fluxions, and so on. Or, if from a towards A we take ar equal to one half of AB, and the ordinate rv meet the curve in v, we may make use of the second series, provided A represent in it the area RrvV, b represent the difference of the fluxions of RV and rv, d the difference of their third fluxions, and so on.

353. When the limit of the progression AF, BE, CK, HL, &c. is given, and the limit of the area is required, let the former be P; and, according to the first supposition, where a expresses AF, and b, d, f, &c. express its first, third, fifth fluxions, &c. the limit of the area is found by computing $P - \frac{1}{2}a - \frac{1}{12}b + \frac{1}{720}d - \frac{1}{30240}f -$ &c. But, according to the second supposition, the limit of the area is found by computing $P + \frac{1}{24}b - \frac{7}{5760}d + \frac{31}{967680}f -$ &c. The first expression approximates to the area AafF when P is supposed to represent the sum of the terms from AF to af, (excluding the latter,) a the difference of AF and af, b the difference of their first fluxions, d the difference of their third fluxions, and so on. And the second series approximates to the area RVvr when a is the difference of RV and rv, b the difference of their first fluxions, d the difference of their third fluxions, and so on. We refer the farther explication of this, with the demonstration and examples, to the second book. We shall now shew how progressions of fractions may be found at pleasure that shall have assignable numbers equal to the limit of the sum of the terms.

354. A series of any number of quantities that continually decrease being given, their successive differences form a new series of terms, the sum of which from the beginning is always equal to the excess of the first term of the first series above its last term. Thus, if A, B, C, D, E, &c. be the terms of the first series, it is manifest that the sum of the differences of A and B, B and C, C and D, D and E, is the excess of A above E. If the terms of the first series decrease in such a manner that by continuing the progression they may become less than any quantity how small soever that can be assigned, (as the ordinates to the asymptote become less than any line that may be given by pro-

producing the figure,) then the first term of the first series is the limit of the sum of the second series. In like manner, the differences of the alternate terms of the first series, as of A and C, B and D, C and E, *&c.* form a new progression of terms, the sum of any number of which is equal to the excess of the sum of A and B, the first and second term of the first series, above the sum of the last term and last but one; and the sum of A and B is the limit of the sum of the new series. In general, if a progression is formed by taking the differences of the first term A and the term whose place in the series is expressed by any number n, of the second term B and that whose place is $n+1$, of the third term C and that whose place is $n+2$, and so on; then the limit of the sum of this new progression shall be equal to the sum of the terms A, B, C, D, *&c.* which preceed that term whose place is expressed by n. In this manner progressions may be found, at pleasure, that may be continued without end, and have given numbers for the limits of their sums: And this method differs not materially from that of the celebrated Mr. JAMES BERNOULLI.

355. For example, let the first series be $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, *&c.* The successive differences of those terms are $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}$, *&c.* and the limit of the sum of this progression is unit, by the last article. If we multiply each term of this last series by 2, (that the first term may be unit,) we shall have $1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}$, *&c.* which have the triangular numbers for their successive denominators, unit being their common numerator; and therefore the limit of the sum of this progression is 2. The successive differences of the terms of this latter series being each multiplied by $\frac{3}{2}$, (that the first term of the new series may be unit,) give $1, \frac{1}{4}, \frac{1}{10}, \frac{1}{20}$, *&c.* which have the pyramidal numbers for their successive denominators; and the limit of the sum of this progression is $\frac{3}{2}$. In the same manner, the limit of the sum of the fractions that have unit for their common numerator, and the figurate numbers of any order denoted by m for their successive denominators, is found to be $\frac{m-1}{m-2}$.

356. The same series $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$, *&c.* being assumed again, the differences of the alternate terms are $\frac{2}{3}, \frac{2}{8}, \frac{2}{15}, \frac{2}{24}$, *&c.* the limit

limit of the ſum of which progreſſion is $1\frac{1}{2}$, by art. 354. and, dividing each term by 2, the limit of the ſum of $\frac{1}{3}$, $\frac{1}{8}$, $\frac{1}{15}$, &c. is $\frac{3}{4}$. If we take the differences of the firſt term and that whoſe place is m, the ſecond term and that whoſe place is $m+1$, and ſo on; the common numerator of thoſe differences ſhall be $m-1$, and their ſucceſſive denominators ſhall be the products of 1 and m, 2 and $m+1$, 3 and $m+2$, and ſo on. The limit of the ſum of this progreſſion is the ſum of as many terms 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. as there are units in $m-1$, by art. 354. and if each term of that progreſſion be divided by $m-1$, that unit may become the common numerator, the terms $\frac{1}{m}$, $\frac{1}{2.m+1}$, $\frac{1}{3.m+2}$, $\frac{1}{4.m+3}$, &c. will ariſe, (where the factors in the denominators that are ſeparated by a point are ſuppoſed to be multiplied by each other,) and the limit of the ſum of this progreſſion is equal to the ſum of the fractions 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c. (continued till their number be $m-1$) divided by $m-1$. If we now aſſume the alternate terms only of the firſt ſeries beginning with the ſecond, that is, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, &c. and form a new ſeries by taking the difference of the firſt of theſe $\frac{1}{2}$ and that whoſe denominator is $m+1$, (m being any odd number,) of $\frac{1}{4}$ and that whoſe denominator is $m+3$, and ſo on; the new terms ſhall have $m-1$ for their common numerator, and the products of 2 and $m+1$, 4 and $m+3$, 6 and $m+5$, &c. for their ſucceſſive denominators: And the limit of the ſum of thoſe fractions ſhall be equal to the ſum of the fractions $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, &c. that preceed $\frac{1}{m+1}$, the number of which fractions is $\frac{m-1}{2}$. By aſſuming the other alternate terms of the firſt ſeries, that is, 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c. and taking the differences of 1 and the term whoſe denominator is m, of $\frac{1}{3}$ and that whoſe denominator is $m+1$, and ſo on, the terms of the new ſeries ſhall have $m-1$ for their common numerator, and the products of 1 and m, 3 and $m+2$, 5 and $m+4$, &c. for their ſucceſſive denominators; and the ſum of the terms ſhall continually approximate to the ſum of the fractions 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, &c. that preceed $\frac{1}{m}$. We may likewiſe aſſume any other equi-

equidistant terms of the first series, and, by taking their differences, form new progressions the value of which shall be the first term that was assumed. If we assume the terms 1, $\frac{1}{5}$, $\frac{1}{9}$, $\frac{1}{13}$, &c. passing always over three terms, and divide the successive differences of those terms by 96, the series will arise that is given by the celebrated Mr. MONMORT in Philos. Transact. n. 353. and is marked C; and therefore the value of this series (when it is supposed to be continued infinitely) is $\frac{1}{96}$, by art. 354. If we assume the alternate terms of the first series deduced in art. 355. *viz.* $\frac{1}{2}$, $\frac{1}{12}$, $\frac{1}{30}$, $\frac{1}{56}$, &c. and divide the successive differences of those terms by 2, the series will arise that is marked B in the same Transaction; the value of which is therefore $\frac{1}{4}$. If we assume the first, fourth, seventh terms, &c. of the same series from art. 355. passing always over two terms, their differences divided by 54 coincide with the series marked A in the same place; and therefore the value of this series is $\frac{1}{108}$. If we multiply the corresponding terms of any two progressions by each other, and if the products may become less than any given number by continuing them, (as for example, if we multiply the successive terms of the series 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c. by the successive powers of any fraction $\frac{1}{n}$,) their differences shall always give progressions of this kind. It is obvious from art. 354. how the sum of any given number of terms is found in the progressions we have mentioned, or in any others that are deduced in this manner. But what follows seems to have a nearer relation to the method of fluxions, and to be of greater use.

357. Let us assume the series $\frac{1}{m}$, $\frac{1}{m+1}$, $\frac{1}{m+2}$, $\frac{1}{m+3}$, &c. where unit is the common numerator, and the denominators increase by the continual addition of unit. The successive differences of those terms are $\frac{1}{m \cdot m+1}$, $\frac{1}{m+1 \cdot m+2}$, $\frac{1}{m+2 \cdot m+3}$, $\frac{1}{m+3 \cdot m+4}$, &c. and the limit of the sum of this series (or its value when it is supposed to be infinitely produced) is $\frac{1}{m}$, the first term of the series which we assumed, by art. 354. The successive differences of

of the terms of this last series divided by 2 are $\frac{1}{m\,.\,\overline{m+1}\,.\,\overline{m+2}}$, $\frac{1}{\overline{m+1}\,.\,\overline{m+2}\,.\,\overline{m+3}}$, $\frac{1}{\overline{m+2}\,.\,\overline{m+3}\,.\,\overline{m+4}}$, &c. and the limit of their sum is therefore $\frac{1}{2m\,.\,\overline{m+1}}$, by the same article. The successive differences of those terms divided each by 3 are $\frac{1}{m\,.\,\overline{m+1}\,.\,\overline{m+2}\,.\,\overline{m+3}}$, $\frac{1}{\overline{m+1}\,.\,\overline{m+2}\,.\,\overline{m+3}\,.\,\overline{m+4}}$, &c. and the limit of their sum is therefore $\frac{1}{3m\,.\,\overline{m+1}\,.\,\overline{m+2}}$. The terms in each of those progressions are formed from the first term by substituting successively in its denominator $m+1$, $m+2$, $m+3$, $m+4$, &c. in place of m: And it appears, that, in general, if we substitute successively m, $m+1$, $m+2$, $m+3$, &c. in place of m in the fraction $\frac{1}{m\,.\,\overline{m+1}\,.\,\overline{m+2}\,.\,\&c.}$, (where the factors in the denominator are supposed to be continued till their number be $n+1$,) the terms that shall be produced in this manner shall form a progression the value of which is found by substituting n, in the denominator of that fraction, in place of the last and greatest factor. This useful theorem has been demonstrated by several eminent Mathematicians, Dr. TAYLOR, Mr. NICOLE, and of late by Mr. STIRLING *, who has much improved the methods of approximating to the values of progressions that arise in the resolution of problems. It is obvious from art. 354. how the sum of any given number of terms may be found in the progressions we have described.

358. The same series $\frac{1}{m}$, $\frac{1}{m+1}$, $\frac{1}{m+2}$, $\frac{1}{m+3}$, - - - $\frac{1}{m+n}$, &c. being assumed, where m is supposed to represent any quantity at pleasure, and n any integer number; let a new progression be formed by taking the differences of the first term $\frac{1}{m}$ and of $\frac{1}{m+n}$, of the second $\frac{1}{m+1}$ and $\frac{1}{m+n+1}$, of the third $\frac{1}{m+2}$ and

* De summatione serierum, prop. 2.

$\frac{1}{m+n+2}$, and so on. Because the difference of the denominators of those fractions is always n, it is manifest that the terms of the new progression shall be $\frac{n}{m \cdot m+n}$, $\frac{n}{m+1 \cdot m+n+1}$, $\frac{n}{m+2 \cdot m+n+2}$, &c. The value of this progression is found by summing up all the terms in the first series that preceed the term $\frac{1}{m+n}$, by art. 354. the number of which terms is n. Hence it appears, that when n is any integer number, if we substitute successively m, $m+1$, $m+2$, $m+3$, &c. in place of m in the fraction $\frac{1}{m \cdot m+n}$, the value of the progression that shall be thus formed is equal to $\frac{1}{n \cdot m} + \frac{1}{n \cdot m+1} + \frac{1}{n \cdot m+2} +$ &c. those terms being continued till their number be n. But when n is a fraction, the value of the progression that is formed in the same manner cannot be assigned accurately in numbers, but we may approximate to it readily by a method explained below, (art. 361.) Several eminent Mathematicians have treated of this subject, besides those already mentioned, as Mr. LEIBNITZ, Mess. BERNOULLI, and Mr. de MOIVRE; and various methods have been given by which an infinite variety of such progressions may be found. The following perhaps may be worth mentioning.

FIG. 147. 359. The right line AB being given, let the point P set out from A towards B, and proceed always in the same direction with an uniform motion; let the point p set out at the same time from B towards A with a velocity greater than that of A; and let it be constantly reflected with an uniform motion betwixt P and the fixed point B, so as to describe BD while P describes AD, and to describe DB + BE, EB + BF, FB + BG, &c. in the same times that P describes DE, EF, FG, &c. respectively: Then the spaces described by P and p betwixt the terms when they meet each other shall form two geometrical progressions, and the common ratio of the terms in both shall be that of the sum of the velocities of P and p to their difference.

rence. If we suppose the velocity of p to increase or decrease every time it comes to B, (but still so as to be greater than the constant velocity of P,) and to continue uniform till it return again to B, the spaces described by P betwixt the terms when it meets p shall form progressions of various kinds, that are easily determined when the rule is given according to which the velocity of p increases or decreases. But to obtain progressions whose terms may have more simple expressions, let us conceive p not to be reflected from P to B, but every time it meets P to be instantly brought back to B, and to set out anew from B towards A till it meet P again, so as to describe BD, BE, BF, BG, &c. in the same times, respectively, that P describes AD, DE, EF, FG, &c. Then the spaces described by P shall form a progression of terms that may be continued without end, when the velocities of P and p (however variable they may be) are always in an assignable ratio to each other while P describes AB; the sum of those terms is always less than AB, but approaches to it as its limit while the progression is continued. Suppose the motion of P to be invariable, and the motion of p to be uniform while P describes any one term; let the constant velocity of P be expressed by m; and the velocities of p, while P describes any two successive terms EF, FG, be expressed by V and u, respectively: then shall FG be to EF as V is to $m+u$, because FG is to BF as m is to $m+u$, and BF is to EF as V is to m. The sum of any number of terms described by P, as AG, is found by subducting from AB a right line that is to the last term FG as u is to m; for such a line is equal to BG.

360. The velocity of P being expressed by m, (as in the last article,) let the successive velocities of p increase equally, and be expressed by $n+1$, $n+2$, $n+3$, $n+4$, &c. and let r be equal to $m+n$. Then, since AD the first term described by P is to AB as m is to $r+1$, if AB represent unit, AD shall be expressed by $\frac{m}{r+1}$. The ratio of V to $m+u$ is successively that of $n+1$ to $r+2$, $n+2$ to $r+3$, $n+3$ to $r+4$, &c. Therefore, supposing (after Sir ISAAC NEWTON's manner) A to express the first term $\frac{m}{r+1}$, B the second term, C the third, and

so on, the terms described by P are $\frac{m}{r+1}$, $\frac{A.n+1}{r+2}$, $\frac{B.n+2}{r+3}$, $\frac{C.n+3}{r+4}$, &c. The limit of the sum of this progression is AB, or unit; and the sum of any number of terms denoted by z, the last term being l, is $1 - \frac{l.z+n}{m}$. For example, if we suppose the first velocity of p to be equal to the constant velocity of P, and the successive velocities of p to be 1, 2, 3, 4, &c. that is, if m be unit, and n vanish, the terms described by p shall be $\frac{1}{1.2}$, $\frac{1}{2.3}$, $\frac{1}{3.4}$, $\frac{1}{4.5}$, &c. the sum of which must be always less than unit, as in art. 355. Any term FG whose place in the series is denoted by z is $\frac{1}{z.z+1}$; and therefore AG, the sum of as many terms from the beginning as there are units in z, is (by what was shewn in the last article) $\frac{z}{z+1}$; and the sum of the same number of fractions, that have unit for their common numerator, and the triangular numbers 1, 3, 6, 10, &c. for their successive denominators, is 2AG, or $\frac{2z}{z+1}$. The rest remaining, if m be equal to 2, the terms described by P shall be $\frac{4}{1.2.3}$, $\frac{4}{2.3.4}$, $\frac{4}{3.4.5}$, &c. FG is expressed by $\frac{4}{z.z+1.z+2}$, AG by $1 - \frac{2}{z+1.z+2}$; and the sum of the same number of fractions, that have unit for their common numerator, and the pyramidal numbers for their successive denominators, is $\frac{3}{2} - \frac{3}{z+1.z+2}$. In the same manner it will appear, that, the successive velocities of p being represented by 1, 2, 3, 4, &c. if the constant velocity of P be expressed by any positive integer number m, the sum of the same number of fractions that have unit for their common numerator, and the figurate numbers of the order $m+2$ for their successive denominators, shall be to AG as $m+1$ is to m, and is therefore equal to $\frac{1}{m}$ multiplied by the

excess

excess of $m+1$ above the fraction $\frac{2 \cdot 3 \cdot 4 \cdot \&c.}{z+1 \cdot z+2 \cdot z+3 \cdot \&c.}$ where the factors are supposed to be continued in the numerator and denominator till the number in each be m; as has been demonstrated by Mr. Bernoulli *de seriebus infinitis*, § 18. & 19.

361. Supposing n still to vanish, or m to be equal to r, the terms described by P are $\frac{m}{m+1}$, $\frac{m}{m+1 \cdot m+2}$, $\frac{2 \cdot m}{m+1 \cdot m+2 \cdot m+3}$, $\frac{2 \cdot 3 \cdot m}{m+1 \cdot m+2 \cdot m+3 \cdot m+4}$, &c. and since the limit of their sum is unit, it follows (dividing by mm) that $\frac{1}{mm}$ is the limit of the sum of $\frac{1}{m \cdot m+1}$, $\frac{1}{m \cdot m+1 \cdot m+2}$, $\frac{1 \cdot 2}{m \cdot m+1 \cdot m+2 \cdot m+3}$, $\frac{1 \cdot 2 \cdot 3}{m \cdot m+1 \cdot m+2 \cdot m+3 \cdot m+4}$, &c. From which, and what was shewn in art. 357. it follows, that we approximate to the sum of the terms which arise when we substitute successively m, $m+1$, $m+2$, $m+3$, &c. in place of m in the fraction $\frac{1}{mm}$, by summing up the series $\frac{1}{m}$, $\frac{1}{2m \cdot m+1}$, $\frac{2}{3m \cdot m+1 \cdot m+2}$, $\frac{2 \cdot 3}{4m \cdot m+1 \cdot m+2 \cdot m+3}$, &c. or $\frac{1}{m}$, $\frac{A}{2 \cdot m+1}$, $\frac{4B}{3 \cdot m+2}$, $\frac{9C}{4 \cdot m+3}$, &c. where A represents the first term $\frac{1}{m}$, B the second term, and so on. Of the use of this, see Mr. Stirling's treatise *de summatione serierum*, p. 28. In general, when n does not vanish, we found that unit is the limit of the sum of the progression $\frac{m}{r+1}$, $\frac{A \cdot n+1}{r+2}$, $\frac{B \cdot n+2}{r+3}$, $\frac{C \cdot n+3}{r+4}$, &c. Therefore, dividing by m and r, $\frac{1}{m \cdot r}$ is the value of the progression $\frac{1}{r \cdot r+1}$, $\frac{n+1}{r \cdot r+1 \cdot r+2}$, $\frac{n+1 \cdot n+2}{r \cdot r+1 \cdot r+2 \cdot r+3}$, &c. And, by art. 357. we shall appproximate to the sum of the terms that arise by substituting successively r, $r+1$, $r+2$, &c. in place of r in the fraction $\frac{1}{r \cdot m}$, or $\frac{1}{r \cdot r-n}$, if we sum up the terms

terms $\frac{1}{r}$, $\frac{n+1}{2r.\overline{r+1}}$, $\frac{\overline{n+1}.\overline{n+2}}{3r.\overline{r+1}.\overline{r+2}}$, &c. or the series $\frac{1}{r}+\frac{A.\overline{n+1}}{2.\overline{r+1}}$ $+\frac{2B.\overline{n+2}}{3.\overline{r+2}}+\frac{3C.\overline{n+3}}{4.\overline{r+3}}+$ &c. When n is any negative fraction equal to $-s$, $\frac{1}{r.\overline{r-n}}$ becomes $\frac{1}{r.\overline{r+s}}$, and the series becomes $\frac{1}{r}+\frac{A.\overline{1-s}}{2.\overline{r+1}}+\frac{2B.\overline{2-s}}{3.\overline{r+2}}+\frac{3C.\overline{3-s}}{4.\overline{r+3}}+$ &c. which therefore approximates continually to the sum of the terms that arise when r, $r+1$, $r+2$, &c. are substituted successively for r in $\frac{1}{r.\overline{r+s}}$; and this agrees with what is shewn in a different manner in the same excellent treatise, p. 12. & 26. where this series is applied for finding readily the value of the progression $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}$ $+\frac{1}{4.5}+\frac{1}{5.6}+$ &c. that was invented by Lord BROUNKER for the quadrature of the hyperbola.

FIG. 148. 362. Let the point P move now in the right line BA produced beyond A, and the velocity of p be always greater than that of P, that it may overtake it; let the motion of P be still uniform, and its velocity be represented by m; and let the successive velocities of p by which it describes any two terms BE, BG (while P describes EF and FG) be V and u, respectively. Then shall FG be to EF as V is to $u-m$, because FG is to BF as m is to $u-m$, and BF to EF as V is to m. The sum of any spaces AD, DE, EF, FG, described by P, added to BA, (which is supposed to represent unit,) is BG, which is to FG the last of those spaces as u is to m; and if FG the last term of the progression be expressed by l, then BG (the sum of the terms including AB) shall be $\frac{lu}{m}$. When the motion of p is likewise uniform, the spaces described by P continually increase in a geometrical progression; the common ratio of the terms is that of the velocity of p to its excess above the velocity of P; AG the sum of any number of terms described by P is found by subducting AB (or unit) from BG, which is to the last term FG, as the velocity of p is to the velocity of P, or as the number that

that expresses the common ratio of the terms is to the same number diminished by unit: And hence the common rule for finding the sum of any number of terms in a geometrical progression may be deduced. When the constant velocity of P is denoted by m, and the successive velocities of p by $r+1$, $r+2$, $r+3$, &c. let n be equal to $r-m$, and the terms described by P shall be $\frac{m}{n+1}$, $\frac{A.\overline{r+1}}{n+2}$, $\frac{B.\overline{r+2}}{n+3}$, &c. where A represents the first term, B the second, and so on. And if we prefix AB, or unit, to those terms, the sum of any number from the beginning to the term whose place is denoted by z shall be to this term as $z+r-1$ is to m. When m is an integer positive number, and r is equal to m, (or n vanishes,) these terms are the figurate numbers of the order denoted by m, the second number of each order (or the first space described by P) being m. In this case, when m is unit, each term is equal to AB, or unit: When m is 2, the terms are in arithmetical progression, the last term is z, and the sum of the terms $\frac{z.\overline{z+1}}{2}$: When m is 3, the terms are the triangular numbers, the last term is $\frac{z.\overline{z+1}}{2}$, and the sum of the terms is $\frac{z.\overline{z+1}.\overline{z+2}}{2.3}$. And, in general, the sum of the figurate numbers of any order m from the first, or unit, to the last, (whose place is supposed to be denoted by z,) is found by multiplying this last by $\frac{z+m-1}{m}$, or by computing $\frac{z.\overline{z+1}.\overline{z+2}.\overline{z+3}.\&c.}{1.2.3.4.\&c.}$, where the factors are supposed to be continued in the numerator and denominator till there be as many in each as there are units in m. We might suppose the velocity of p to observe other rules, or the velocity of P to vary likewise; but, not to insist further on this subject here, we shall only add, that there are other methods besides that described in art. 350. & 351. by which it may be known when a progression of fractions has a limit, or not; such is the rule given by an author we have often mentioned, That when A, B, C are any successive terms of the progression at a sufficient distance

ſtance from the beginning, and the ratio of A to C is leſs than the ratio of the difference of A and B to the difference of B and C, the progreſſion may be continued till its ſum exceed any given number; but when it is otherwiſe, the ſum of the progreſſion has a limit.

CHAP. XI.

Of the Curvature of Lines, its Variation, and the different kinds of Contact; of the Curve and Circle of Curvature, the Cauſtics by Reflexion and Refraction, the centripetal Forces, and other Problems that have a dependence upon the Curvature of Lines.

363. ANy two right lines applied upon each other, perfectly coincide; and the rectitude of lines admits of no variety. Arches of equal circles applied upon each other, perfectly coincide likewiſe; and the curvature is uniform in all the parts of the ſame, or of equal circles. Arches of unequal circles cannot be applied upon each other ſo as to coincide; but when they touch each other, the arch of the greater circle is leſs inflected from the common tangent, and paſſes betwixt it and the arch of the leſſer circle through the angle of contact formed by them, and is therefore leſs curve. Any two arches of curve lines touch each other when the ſame right line is the tangent of both at the ſame point; but when they are applied upon each other in this manner, they never perfectly coincide, unleſs they are ſimilar arches of equal and ſimilar figures: and the curvature of lines admits of indefinite variety. Becauſe the curvature is uniform in a given circle, and may be varied at pleaſure in them by enlarging or diminiſhing their diameters, the flexure or curvature of circles ſerves for meaſuring that of other lines. There is but one right line that can be the tangent of a given arch of a curve at the ſame point; but circles of an indefinite variety touch it there; and theſe have various degrees of more and leſs intimate contact with it.

364. As

364. As of all the right lines that can be drawn through a given point in the arch of a curve, that is the *tangent* which touches the arch so closely that no right line can be drawn between them, (art. 181.) so of all the circles that touch a curve in any given point, that is said to have the *same curvature* with it, which touches it so closely that no circle can be drawn through the point of contact between them, all other circles passing either within or without them both. This circle is called the *circle of curvature*, its center the *center of curvature*, and its semidiameter the *ray of curvature*, belonging to the point of contact. The arch of this circle cannot coincide with the arch of the curve, but it is sufficient to denote it the circle of curvature that no other circle can pass between them; as the tangent of the arch of a curve cannot coincide with it, but is applied to it so that no right line can be drawn between them. As in all figures, rectilineal ones excepted, the position of the tangent is continually varying; so the curvature is continually varying in all curvilineal figures, the circle only excepted. As the curve is separated from its tangent by its flexure or curvature, so it is separated from the circle of curvature in consequence of the increase or decrease of its curvature: and as its curvature is the greater or less according as it is more or less inflected from the tangent, so the variation of curvature is the greater or less according as it is more or less separated from the circle of curvature. It is manifest, that there is but one circle of curvature belonging to an arch of a curve at the same point; for if there were two such circles, any circles described between these through that point would pass between the curve and circle of curvature, against the supposition.

365. When any two curve lines touch each other in such a manner that no circle can pass between them, they must have the same curvature, by the last article; for the circle that touches the one so closely that no circle can pass between them, must touch the other in the same manner. It will appear from the following proposition, that circles may touch curve lines in this manner, that there may be indefinite degrees of more or less intimate contact between the curve and the circle of curvature, and that a conic section may be described that shall have the

same curvature with a given line at a given point, and the same variation of curvature, or a contact of the same kind with the circle of curvature.

PROP. XXXII.

366. *Let any curve* EMH *and a circle* ERB *touch the right line* ET *on the same side at* E; *let any right line* TK *parallel to the chord* EB *meet the tangent in* T, EMH *in* M, *and a curve* BKF *that passes*
Fig. 149. & 150. *through* B *in* K. *Then, if the rectangle* MTK *be always equal to the square of* ET, *the curvature of* EMH *at* E *shall be the same as that of the circle* ERB; *and the contact of* EM *and* ER *shall be always the closer the less the angle is that is contained at* B *by the curve* BKF *and the circle of curvature* BQE.

Let the right line TK meet the circle in R and in Q, and, since the rectangle RTQ is equal to the square of ET, it must be equal to the rectangle MTK, by the supposition; and there-
Fig. 149. fore RT is to MT as TK is to TQ. Suppose first that BK the part of the curve BKF that is next to the point B, adjoining to it, falls without the circle BQ; and suppose the right line TK by moving parallel to itself to approach to EB till it coincide with it; and while the point K describes KB, TK being greater than TQ, RT must be greater than MT, and the arch EM of the curve must pass without the circle ER betwixt it and the tangent ET: And since any circle described through E, upon a chord less than EB touching ET, falls within the circle ERB, it is manifest that no such circle can pass betwixt the curve EM and circle ERB. Let any circle E*rb*, described upon a chord E*b* greater than EB, touch ET, and meet TK in *r* and *q*; and, since the rectangle *r*T*q* is equal to the square of ET, or the rectangle MTK, MT is to *r*T as T*q* is to TK: And, since the curve FKB passes through B (by the supposition) so that the part of it that is next adjoining to B must be within the arch *bq* of the circle *bq*E, it follows that while K describes this

this part of FKB, Tq must be greater than TK, and, consequently, MT greater than rT. Therefore the arch Er of the circle Erb is without the curve EM, and passes betwixt it and the tangent ET. Therefore no circle whatever can pass betwixt EM the arch of the curve and ER the arch of the circle; and, consequently, the circle ERB has the same curvature with EM at E. Suppose now that the part of the curve BKF that is next adjoining to B falls within BQ the arch of the circle BQE; then, while K describes this part of the curve FKB, TK being less than TQ, RT must be less than MT, and the arch EM of the curve must fall within ER the arch of the circle; and, since any circle described through E upon a chord greater than EB falls without the circle ER, it is manifest that no such circle can pass betwixt ER and EM. Nor can any circle Erb, described upon a chord Eb less than EB, touching ET, pass between ER and EM: for let TK meet this circle in r and q, and MT being to rT as Tq is to TK, and Tq being less than TK while K describes the part of the curve FKB that is next adjoining to B, MT must be less than rT; and, consequently, the arch Er of the circle Erb must fall within EM the arch of the curve. Therefore, in either case, all the circles that can be described through E fall without both ER and EM, or within them both, and no circle whatever can pass between them, when the rectangle MTK is always equal to the square of ET, and the curve in which K is always found passes through B; that is, the circle ERB and the curve EM have the same curvature at E, by art. 364. which was the first part of the proposition. FIG. 150.

367. Let Em any other curve touching ET in E, and fkB another curve passing through B, meet TK in m and k, and let the rectangle mTk be likewise equal to the square of ET; then the curvature of Em at E shall be the same as that of the circle ERB, or that of the curve EM, by what has been demonstrated. Because the rectangles mTk, MTK, RTQ are equal to each other, Tm is to TM as TK is to Tk, and Tm to TR as TQ is to Tk. Therefore, if the arch Bk pass between BK, the arch of the curve BKF, and BQ the arch of the circle BQE, the curve Em must pass between EM, the arch of the curve FIG. 151.

EMH, and ER the arch of the circle of curvature ERB; so that E*m* must have a closer contact with this circle than EM has with it: And the less the angle is that is formed by the curve FKB and the circle of curvature EQB at B, the closer is the contact at E of the curve EMH and the circle of curvature ERB. Thus the curve BKF by its intersection with EB determines the curvature of EM, and by the angle in which it cuts the circle of curvature it determines the degree of contact of EM and that circle, the angle BET and right line ET being given.

368. Cor. I. It appears from the demonstration, that according as the arch BK of the curve BKF falls without, or within, the arch BQ of the circle BQE, the arch EM of the curve EMH falls without, or within, the circle ERB; that when the curve FKB cuts the circle ERB in B, the curve HME cuts the circle of curvature in E; that when the curve FKB is on the same side of the circle BQE on both sides of B, the curve HME continued on both sides of E is on the same side of the circle of curvature; and that the contact of the curve EMH and the circle of curvature is closest when the curve BK touches the arch BQ in B, the angle BET being given, but it is farthest from this, or is most open, when BK touches the right line EB in B.

369. Cor. II. There may be indefinite degrees of more and more intimate contact between a circle ERB and a curve EMH. The first degree is, when the same right line touches them both in the same point; and a contact of this sort may take place betwixt any circle and any arch of any curve. The second is, when the curve EMH and circle ERB have the same curvature, and the tangents of the curve BKF and circle BQE intersect each other at B in any assignable angle. The contact of the curve EM and circle of curvature ER at E is of the third degree, or order, and their osculation is of the second, when the curve BKF touches the circle BQE at B, but so as not to have the same curvature with it. The contact is of the fourth degree, or order, and their osculation of the third, when the curve BKF has the same curvature with the circle BQE at B, but so as that their contact is only of the second degree: And this grada-

gradation of more and more intimate contact, or of approximation towards coincidence, may be continued indefinitely, the contact of EM and EK at E being always of an order two degrees closer than that of BK and BQ at B. There is however an indefinite variety comprehended under each order. Thus, when EM and ER have the same curvature, the angle formed by the tangents of BK and BQ admits of indefinite variety, and the contact of EM and ER is the closer the less that angle is: And when that angle is of the same magnitude, the contact of EM and ER is the closer the greater the circle of curvature is: for since TR is to TM as TK is to TQ, RM (which subtends the angle of contact MER) is to TR as KQ is to TK; and, consequently, RM is to KQ as the square of ET is to the rectangle KTQ; so that, when ET is given, RM is as KQ directly and the rectangle KTQ inversely; and when KQ is given, RM is less in proportion as the rectangle KTQ is greater. When BK touches the circle BQ at B, they may touch it on the same, or on different sides of their common tangent; and the angle of contact KBQ may admit of the same variety with the angle of contact MER in the former case. But as there is seldom occasion for considering those higher degrees of more intimate contact of the curve EMH and circle of curvature ERB, we shall call their *contact* or *osculation of the same kind*, when, the chord EB and angle BET being given, the angle contained by the tangents of BK and BQ is of the same magnitude.

369. Cor. III. The curvature is uniform in the circle only. When the curvature of EMH increases from E towards H, and consequently corresponds to that of a circle gradually less and less, the arch EM falls within ER the arch of the circle of curvature, and BK is within BQ. When the curvature of EM decreases from E towards H, and consequently corresponds to that of a circle that is gradually greater and greater, the arch EM falls without ER the arch of the circle of curvature; and BK is without BQ. According as the curvature of EM varies more or less, it is more or less unlike to the uniform curvature of a circle; the arch of the curve EMH separates more or less from the arch of the circle of curvature ERB, and the angle con-

contained by the tangents of BKF and BQE at B is greater or less. Thus the *quality* of curvature (as it is called by Sir ISAAC NEWTON in a treatise lately published by the ingenious Mr. COLSON) depends on the angle contained by the tangents of BK and BQ at B; and the measure of the inequability or variation of curvature is as the tangent of this angle, the radius being given, and the angle BET being right; for the index of this variation is as the fluxion of the ray of curvature directly and the fluxion of the curve inversely: and we shall shew afterwards (art. 386.) that the fluxion of TK when M comes to E is always to the fluxion of the ray of curvature in the invariable ratio of two to three. The measure of the angle of contact MER, contained by the curve and circle of curvature, depends not on the angle in which BK intersects BQ, only; but is as the tangent of this angle directly and the square of the ray of curvature inversely (as will appear afterwards) when the tangents of BK and BQ intersect each other in any assignable angle; and this measure observes other proportions when BK touches BQ at B.

FIG. 152. 370. COR. IV. The rays of curvature of similar arches in similar figures are in the same ratio as any homologous lines of these figures, and the curve BK cuts BQ in the same angle, or the variation of curvature is the same. For let EM and *em* be any similar arches situated as we described in art. 122. so that, S being a given point, if SM meet EM and *em* in M and *m*, SM may be always to S*m* in the same ratio as SE is to S*e*. Let SM and S*m* meet the tangents ET, *et* (which are parallel to each other because the figures are similar and similarly situated) in Z and *z*; and MZ shall be to *mz*, TM to *tm*, and ET to *et*, as SE is to S*e*. Therefore, since the rectangle MTK is to *mtk* as the square of ET is to the square of *et*, TK is to *tk*, MK to *mk*, and SK to S*k*, as SE is to S*e*; and the figures SKB, S*kb* are likewise similar by art. 122. Therefore EB is to *eb* as SE is to S*e*; and (because the angles BET, *bet* are equal) the rays of curvature are in the same ratio. The tangents of BK and *bk* at B and *b* are parallel, and cut the circles of curvature BQ and *bq* in equal angles: Therefore the variation of curvature is the same in similar arches. When the tangent of BK cuts the tangent

of

Pl. XVI. Pa. 310.

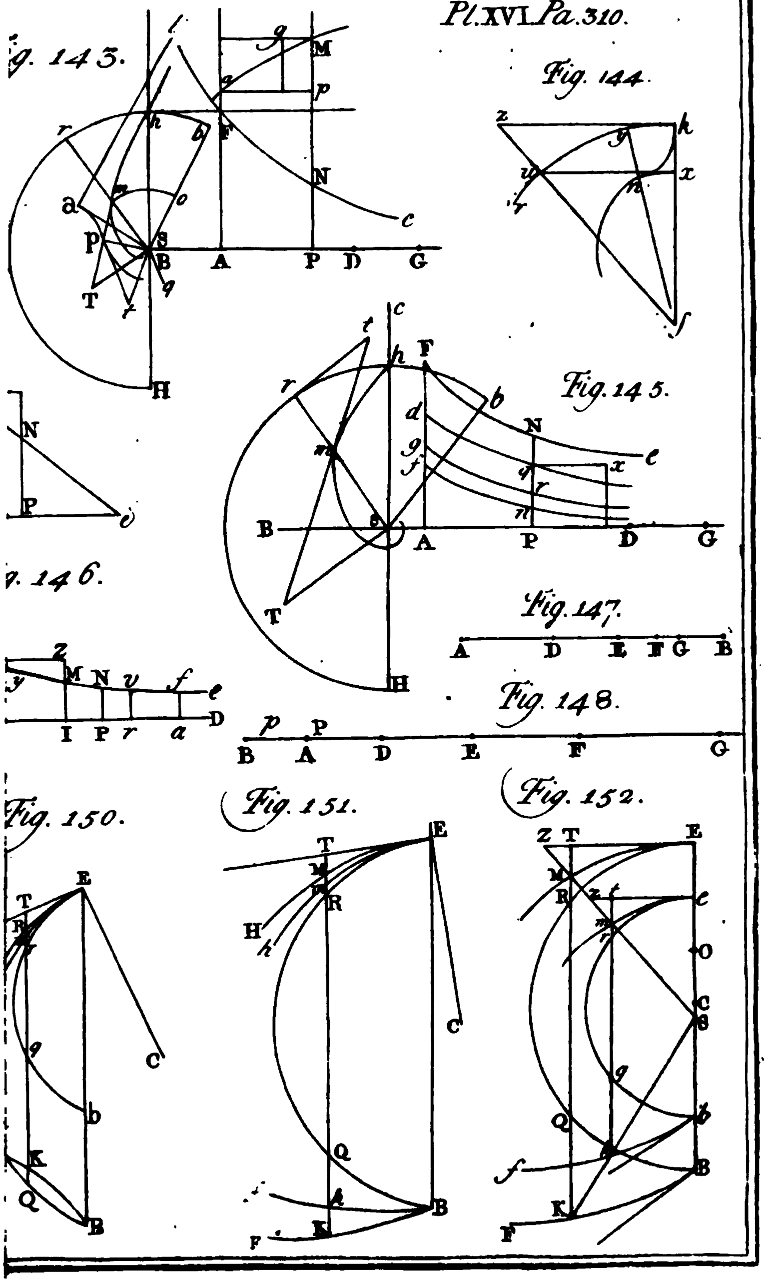


Fig. 143.
Fig. 144.
Fig. 145.
Fig. 146.
Fig. 147.
Fig. 148.
Fig. 150.
Fig. 151.
Fig. 152.

of BQ in any assignable angle, the measures of the angles of contact MER and *mcr* contained by similar arches and their circles of curvature, are reciprocally as the squares of the rays of curvature, or of any homologous lines of the figures. When BK touches BQ, but so as not to have the same curvature with it, those measures are reciprocally as the cubes of the rays of curvature. When BK touches BQ and has the same curvature with it, and their contact is of the second degree only; then those measures are reciprocally as the fourth powers of the rays of curvature, and so on, as will appear afterwards.

371. Cor. V. Let the curve EMH, for example, be a parabola, EB a diameter, ET the tangent at E; and, because the rectangle contained by TM and the parameter of the diameter EB is equal to the square of ET, or the rectangle MTK, it follows, that TK is always equal to this parameter; that in this case BK is a right line parallel to the tangent ET, and that it intersects EB in B so that EB is equal to that parameter. Therefore, if upon the diameter of a parabola a right line EB be taken from E the vertex of this diameter equal to its parameter, a circle ERB described upon this right line as its chord, that touches the parabola at E, shall be the circle of curvature. Because the right line BK cuts the circle BQE in B, unless when E is the vertex of the figure, the parabola cuts the circle of curvature (that case excepted) and passes within the circle of curvature when it is produced towards the vertex, but without it when produced the contrary way. When a parabola EMH and any curve E*mb* have the same curvature at E, no parabola can be drawn through E betwixt EMH and E*mb*; for supposing the rectangle *m*T*k* to be always equal to the square of ET, the curve *fk* shall pass through B, (by the supposition:) and it is shewn, in the very same manner that this proposition was demonstrated, that any other parabola described through E shall either fall without both EM and E*m*, or within them both. Hence the chord of curvature EB of any curve E*mb* is found, when the parabola is determined that touches it so closely that no parabola can pass between them. Fig. 153.

372. Cor. VI. Let EM be an hyperbola, AP and AH the asymptotes; let the tangent ET meet the asymptote AP in V; Fig. 154.

let the angle EVB be made equal to the angle HAP contained by the aſymptotes, and VB meet EB parallel to the aſymptote within the curve in B: then a circle deſcribed upon the chord EB that touches ET ſhall be the circle of curvature. For, in this caſe, the point K is always found in the right line VB; becauſe, if MT produced meet EP parallel to AH in Z, and TG parallel to VB meet EB in G, then, becauſe of the ſimilar triangles EZT, ETG, the rectangle contained by EG and ZT ſhall be equal to the ſquare of ET, or the rectangle MTK; and TK ſhall be to EG as ZT is to TM, or (by the properties of the hyperbola) as PZ is to EZ, or VT to TE, or GB to EG; therefore TK is equal to GB, and K is always found in the right line VB. Becauſe VB cuts the circle BQ, unleſs when the angles EBV and BEV are equal and conſequently E is the vertex of the figure, it follows that the hyperbola cuts the circle of curvature in all other caſes. When ET the tangent of any curve EM and BV the tangent of the curve BK are given, an hyperbola may be deſcribed through E, that ſhall have the ſame curvature with EM and the ſame variation of curvature, or a contact with the circle of curvature of the ſame kind, by producing thoſe tangents till they meet in V, and deſcribing through E an hyperbola that has an aſymptote through V parallel to EB, and another aſymptote through *u* (E*u* being taken equal to EV upon VE produced beyond E) that conſtitutes with the former an angle *u*AV equal to EVB, or that makes the angle E*u*A equal to EBV.

FIG. 155. 373. COR. VII. Let EMH be any conic ſection, ET the tangent at E, HI a tangent parallel to EB that meets ET in I, and let EMH meet EB in G. Take EB to EG in the ſame ratio as the ſquare of EI is to the ſquare of HI, or (when the conic ſection has a center) as the ſquare of the ſemidiameter O*a* parallel to ET is to the ſquare of the ſemidiameter OA parallel to EB, and a circle deſcribed upon the chord EB that touches ET ſhall be the circle of curvature. For let TM meet the conic ſection again in *m*; and, ſince the rectangle MT*m* is to the ſquare of ET, or the rectangle MTK, as the ſquare of HI is to the ſquare of EI, T*m* is always to TK in the ſame ratio; and, conſequently, the point K is always found in a conic ſe-

ction,

ction, that meets EG in B so that EB is to EG as the square of EI is to the square of HI. Let GV the tangent of the first conic section EMG at G meet ET in V; and, because T*m* is to TK in an invariable ratio, and their fluxions are in the same ratio, it follows from prop. 14. that BV shall be the tangent of the conic section BKF at B. Hence it easily appears whether the arch EM of the conic section falls without or within the circle of curvature, and the different degrees of osculation betwixt the conic section and the circle of the same curvature may be compared together. For let any conic section EMH meet the chord of curvature EB in G, draw the tangent GV meeting ET in V, join BV, and, the angle BET being given, the nearer the angle EBV is to BET, or the nearer the ratio of EV to BV is to a ratio of equality, the closer is the contact of the conic section EM and of the circle of curvature at E. Let the angle EB*v* be made equal to BET, and let B*v* meet ET in *v*, join G*v*, and of all the conic sections which can be described through E and G having the same curvature at E with the circle ERB, that which touches the right line G*v* has the closest contact with the circle: When BET is a right angle, or EB is the diameter of the circle of curvature, EG is in this case the axis of the conic section (because the angle EB*v* is a right one,) EB is the parameter of this axis, and when the points G and B are on the same side of E, EMG is an ellipse, and EG is the greater or the lesser axis according as EG is greater or less than EB.

374. Cor. VIII. The propositions relating to the curvature of the conic sections, which are delivered by authors on this subject, follow from what has been demonstrated. 1. When the chord of curvature EB passes through O the center of the conic section, A coincides with E, EG is a diameter, OA and O*a* are conjugate semidiameters, and, since EB is to EG as the square of O*a* is to the square of OA (or OE,) or as the parameter of the diameter EG is to EG, it follows, that EB the chord of the circle of curvature is equal to the parameter of EG the diameter that passes through the point of contact. This was shewn of the parabola, in art. 371. 2. Let C be the center of the circle of curvature, and C*b* be perpendicular on the dia- Fig. 156.

diameter EG in *b*; let O*a* meet EC in N: and, the triangles EON, EC*b* being similar, the rectangle CEN is equal to the rectangle OE*b*, which is equal to the square of O*a*, because E*b* is one half of the parameter of EG. Therefore the square of the semidiameter O*a* is to the rectangle contained by O*a* and EN, or the invariable rectangle contained by half the transverse and half the conjugate axis, as the ray of curvature CE is to O*a*; and the cube of the semidiameter O*a*, that is conjugate to OE which passes through the point of contact, is equal to the solid contained by the ray of curvature and that invariable rectangle, as is shewn by Mr. de Moivre, *Misc. analyt.* p. 235. 3. Let EH be an ordinate to an axis of the conic section, and, EI being in this case equal to HI, EB shall be equal to EG, or B coincide with G: from which it follows, that if EH be drawn from E perpendicular to either axis, and the angle HEG be made equal to HET on the opposite side of HE, then EG shall meet the conic section in the point G where the circle of curvature and conic section intersect each other.

Fig. 157.

Fig. 158. n. 1. & 2.

375. Cor. IX. To these we may add the following properties of the circle of curvature belonging to any point of a conic section. 4. In the ellipse or hyperbola, let O*a*, the semidiameter parallel to ET the tangent of the section at E, meet EB, any chord of the circle of curvature, in R; bisect EB in *b*; and the rectangle RE*b* shall be equal to the square of O*a*. For, let the diameter through the point of contact meet the circle of curvature again in *e*, join B*e*; and, the angle E*e*B being equal to BET or ERO, the triangles EB*e*, EOR are similar, EB is to E*e* as EO is to ER, and the rectangle REB is equal to the rectangle OE*e*. But E*e* is equal to the parameter of the diameter through E, by the first property of the circle of curvature in the last article: therefore the rectangle OE*e* (or REB) is equal to twice the square of O*a*, and the rectangle RE*b* is equal to the square of O*a*. When EB passes through the focus, ER is equal to half the transverse axis; therefore the chord of the circle of curvature that passes through the focus, the diameter conjugate to that which passes through the point of contact, and the transverse axis of the figure are in continued proportion. It appears likewise, that when the section is an ellipse, if the circle

circle of curvature at E meet O*a* in *d*, the ſquare of E*d* ſhall be equal to twice the ſquare of O*a*; therefore E*d* is to O*a* in the invariable ratio of the diagonal of a ſquare to its ſide, or of the ſquare root of two to unit: and hence, when the point of contact E and the ſemidiameter O*a* are given in poſition and magnitude, the center of curvature is readily determined. 5. The right line EG being ſuppoſed to meet the conic ſection in any two points E and G, and the tangents at thoſe points to interſect each other in V, let EB be biſected in *b*, join V*b*; then the angle EV*b* ſhall be equal to the angle GEO, or to its ſupplement to two right angles; and a circle through E, V and *b* ſhall always touch the diameter that paſſes through E. For, in the ellipſe and hyperbola, if O*p* parallel to EG meet the tangent EV in *p*, the rectangle contained by VE and E*p* (or RO) ſhall be equal to the ſquare of O*a*, or to the rectangle RE*b*; and EV is to E*b* as RE is to RO. Therefore, ſince OR is parallel to EV, the triangles REO, *b*EV are ſimilar, the angle EV*b* is equal to REO, and a circle through E, V and *b* touches EO. In the parabola, let EG be biſected in *g*; and, ſince E*b* is to E*g* as the ſquare of EI is to the ſquare of HI, (by art. 373.) or as the ſquare of EV is to the ſquare of E*g*; it follows, that the rectangle *b*E*g* is equal to the ſquare of EV, and the angle EV*b* equal to E*g*V or GEO. Hence when, in any conic ſection, the tangents EV, GV are given, and the diameter through E is given in poſition, the point *b* and the center of curvature are readily determined. When any two points in a conic ſection, as E and G, and the tangents at theſe points EV and GV, with the circle of curvature belonging to one of them, as E, are given, the ſection is determined by biſecting EB and EG in *b* and *g*, joining V*b* and V*g*, and making the angle *b*EL equal to *b*VE ſo that EV and EL may be on oppoſite ſides of E*b*: for EL ſhall be a diameter of the ſection, and if it interſect V*g* in O, then O ſhall be the center of the figure; but if EL be parallel to V*g*, the figure is a parabola. When *b* and *g* are on different ſides of E, the figure is an hyperbola; and when theſe points are on the ſame ſide of E, it is an hyperbola or ellipſe according as the angle EV*b* is greater or leſs than E*g*V. When two points E and G of a conic ſection are given, with

FIG. 158. n. 3.

FIG. 158. n. 1. 2. & 3.

with EV the tangent at E, and the points B and D where the circles of curvature at E and G meet EG, the tangent GV is determined, by drawing from G to the tangent EV a right line GV so that GV may be to EV in the subduplicate ratio of GD
FIG. 155. to EB. For if HI the tangent at H meet GV the tangent at G in *i*, H*i* and HI shall be equal; and if the circle of curvature at G meet EG in D, GD shall be to EB as the square of G*i* is to the square of EI, (by art. 373.) or as the square of GV
FIG. 159. is to the square of EV. 6. Let EB be the chord of the circle
n. 1. & 2. of curvature that passes through S the focus of the conic section; let BX parallel to the tangent ET meet EX perpendicular to ET in X; and if XZ be perpendicular to EB in Z, then shall EZ be equal to the parameter of the transverse axis of the figure. For let EK be equal to the half of this parameter; then, since EB is to EZ as the square of EB to the square of EX, or as the square of ER (which in this case is equal to half the transverse axis of the figure), to the square of EN, and, consequently, as the square of O*a* to the square of half the conjugate axis, or (by the fourth property of the circle of curvature) as the rectangle *b*ER is to the rectangle KER, that is, as *b*E is to KE; and, since EB is equal to 2*b*E, it follows, that
FIG. 158. EZ is equal to 2EK. In the parabola, let SP perpendicular on
n. 3. the tangent at E meet it in P, and A be the vertex of the figure; then EB shall be to EZ as the square of SE to the square of SP, or as SE is to SA; and since EB is equal to 4SE, (art.
FIG. 159. 371.) EZ is equal to 4SA. Hence, when the focus S, point
n. 1. & 2. of contact E, the tangent ET, and EB the chord of the circle of curvature through S are given, the principal parameter of the figure is readily determined; or, when the rest are given, the center of curvature is easily found. 7. Let S and *f* be the two *foci* when the section is an ellipse or hyperbola; let ON parallel to the tangent ET from the center O meet EN perpendicular to ET in N; let C be the center of the circle of curvature at E; join SC and *f*N: then shall the angle ESC be equal to EN*f*. For the rectangle CEN is equal to the square of O*a*, (by the fourth property,) the square of O*a* is equal to the rectangle contained by SE and *f*E; therefore CE is to SE as *f*E is to EN: and, since the angle SEC is equal to *f*EN, the triangles

angles SEC, NEf are similar, and the angle ESC is equal to NEf. Hence the center of curvature at E is readily found when the two *foci* are given. 8. If EQ be taken upon EB, that chord of the circle of curvature which passes through the focus S, equal to one fourth part of EB; then SQ, SE and the transverse axis of the section shall be in continued proportion. For, since the rectangle REb is equal to the rectangle contained by SE and fE, SE is to EQ (or one half of Eb) as 2ER is to fE; from which it follows, that SQ is to SE as SE is to 2ER, which is equal to the transverse axis of the figure. Hence, when the focus S, the point of contact E and tangent ET with EB that chord of the circle of curvature which passes the focus are given, let EQ be one fourth part of EB, draw SP perpendicular to ET in P, let Pr bisect SE in r, let PO be taken upon Pr (the same or the coutrary way from P with Pr according as EQ is less or greater than ES) equal to one half of the third proportional to SQ and SE; and O shall be the center of the figure. According as EQ is greater or less than ES the figure is an hyperbola or ellipse; and, when EQ is equal to ES, it is a parabola, the vertex of which is determined by making the angle PSA equal to PSE and PA perpendicular to SA in A. FIG. 158. n. 3.

376. COR. X. The variation of curvature at any point of a conic section, is always as the tangent of the angle contained by the diameter that passes through the point of contact, and the perpendicular to the curve at the same point. Thus the variation of curvature at E is as the tangent of the angle GEO, EG being a perpendicular to the curve, and EO a diameter. For if the tangent at G intersect the tangent ET in V, and b be the center of curvature, the angle EVb shall be equal to GEO, or to its supplement to two right angles, by the fifth property of the circle of curvature demonstrated in the last article. The variation of curvature at E is as the tangent of the angle EVB, or of EVb, (because the tangent of the latter angle is always one half of the tangent of the former;) and, consequently, as the tangent of the angle GEO. Hence the variation of curvature vanishes at the extremities of either axis, and is greatest when the acute angle contained by the diameter OE and the tangent ET is least. When the section is a parabola and S is the focus, FIG. 160. FIG. 161.

focus, the angle GEQ is equal to GES; and the variation is as the tangent of the angle contained by the right line drawn from the point of contact to the focus and the perpendicular to the curve. Hence, when the point of contact E, the tangent EV, with the curvature at E and its variation are given, the parabola is determined: For, BE and EV being given, bisect EB in *b*, join V*b*, make the angle BEb equal to *b*VE, and let Eb meet the circle of curvatnre in b; take ES upon Eb equal to one fourth part of Eb: and S shall be the focus of the parabola. If EB meet the axis of the parabola in N, and EL be perpendicular to the axis in L, the variation of curvature shall be as the tangent of the angle ENL, or as the ordinate EL directly and the parameter of the axis inversely. The measure of the angle of contact contained by the parabola and circle of curvature at any point E, is as EL the ordinate to the axis directly and the cube of SE the distance of E from the focus inversely.

Fig. 162. 377. Cor. XI. When EB does not meet with the curve FK, but is its asymptote, any circle being described touching ET in E, a greater circle shall always pass between it and the curve EM; and the greater this circle is, the closer shall its contact be with the curve EM. For, since the curve FK produced passes without any circle EQB how great soever that can be described through E, EM must always pass betwixt ER and the tangent ET. This is the case in which the curvature is said to be infinitely little (being less than that of any circle) or the ray of curvature infinitely great. The point E in the curve HME is a point of contrary flexure when the two hyperbolic branches of the curve FK proceed along the asymptote EB on different sides of it and with opposite directions, as in the common hyperbola. But when these branches are on the same side of the asymptote and proceed along it with opposite directions, E is a cuspid of the first kind; when they proceed with the same direction on different sides of the asymptote, the curve EM has its concavity turned the same way on both sides of E; when those branches proceed along the asymptote on the same side of it and with the same direction, E is a cuspid of the second kind: and in all those cases the curvature at E is less than that of any circle. Of the first case we have an example in the vertex

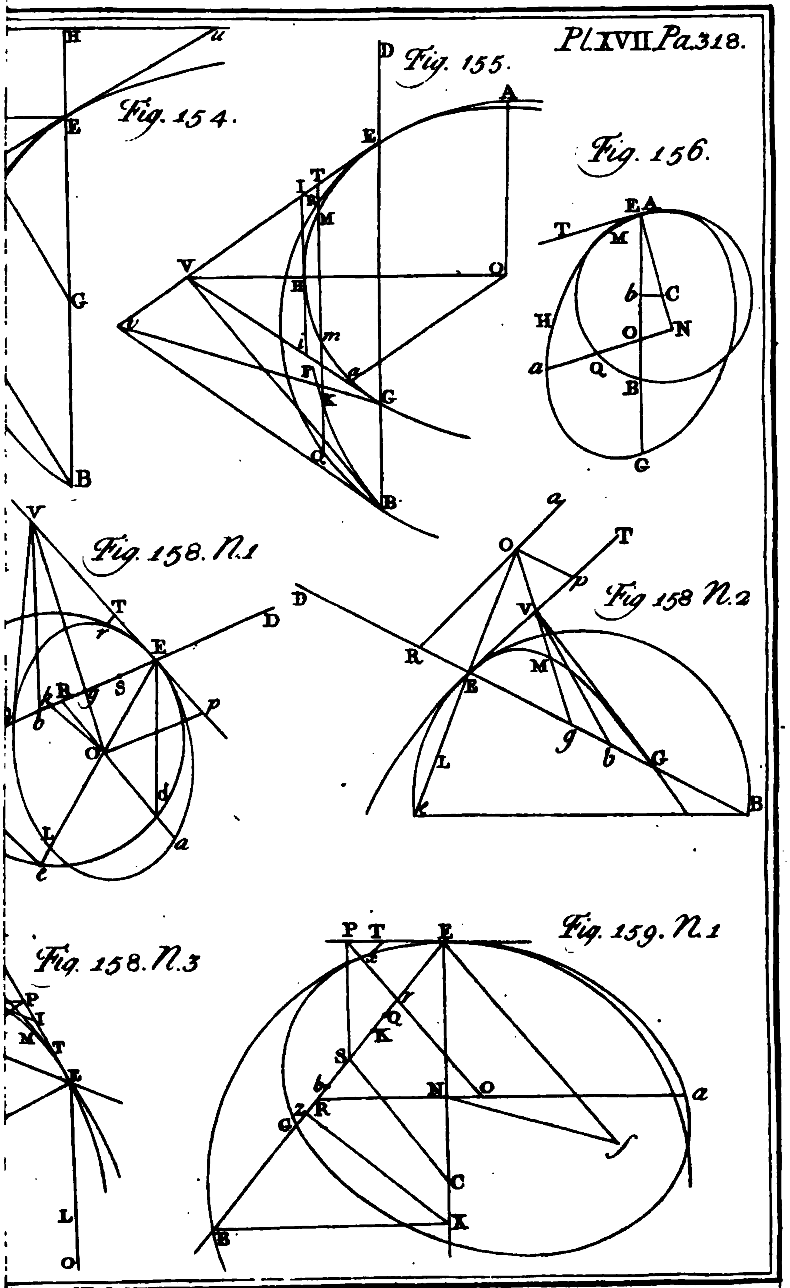


Fig. 154.
Fig. 155.
Fig. 156.
Fig. 158. N. 1
Fig. 158 N. 2
Fig. 158. N. 3
Fig. 159. N. 1

vertex of the cubic parabola, wherein the cube of ET being equal to the solid contained by TM and a given square, the rectangle contained by TK and ET must be equal to this given square, and the curve FK is a common hyperbola that has EB and ET for its asymptotes. The curvature is of the same kind at the vertex of any parabola wherein TM is as any power of ET whose exponent exceeds 2, for FK in all those cases is an hyperbola of which EB is an asymptote.

378. Cor. XII. When the curve FK passes through E, no Fig. 162. circle can be described through E so small but a less circle shall pass between it and the curve EM; and the less this circle is, the closer shall its contact with EM be. For, since the curve FK passes within any circle that can be described through E on the same side of ET, the arch EM of the curve EMH is always within ER the arch of any circle ERB. In this case, because the curvature surpasses that of any circle, it is said to be infinitely great, or the ray of curvature to be infinitely little. When FK passes through E, it always touches EB there, and if it has a continued curvature at E, the curve HME has a cuspid of the first kind at E; if FK has a point of contrary flexure at E of any kind, HME has likewise a point of contrary flexure at E; if FK has a cuspid of the first kind at E, HME has its concavity turned the same way on both sides of E; and if FK have a cuspid of the second kind at E, HME shall have a cuspid of the same kind there; and in all those cases the curvature of HME at E surpasses that of any circle, of whatever kind the curvature of FKE at E be. Of the first, we have an example at the vertex or cuspid of the semicubical parabola, in which the cube of ET being equal to the solid contained by the square of TM and an invariable right line, the square of TK is equal to the rectangle contained by ET and the same invariable right line; and therefore FK is a common parabola that touches EB in E. When FKB intersects EB in two points neither of which coincide with E, then E is not a cuspid, but a point where two arches of the curve HME touch each other that are both continued on each side of E. Of this kind there are no points in the lines of any order beneath the fourth, as there are no points of con-

contrary flexure, or cuspids, in the lines of any order beneath the third.

FIG. 163. 379. COR. XIII. Suppose now that EMH is any line of the third order; let ET the tangent at E meet the curve in A, EG any right line through E meet it in the points E, G and *g*, and TM parallel to EG meet it in M, *m* and m: then, if EB be taken upon EG on the concave side of the arch EM, so that the rectangle AEB be to the rectangle GE*g* as the solid contained by AT and the square of ET is to the solid contained by TM, T*m* and Tm, a circle described upon the chord EB so as to touch ET shall be the circle of curvature at E. For these solids are in an invariable ratio to each other when ET and EG are given in position and TM is alwas parallel to EG, by the properties of the lines of the third order; and, since the rectangle MTK is supposed equal to the square of ET, it follows, that the rectangle ATK is to the rectangle *m*Tm in the same invariable ratio: Therefore the curve FKB shall meet EG in B so that the rectangle AEB shall be to GE*g* in the same ratio. In all the lines of this order the curvature at the confine of contrary flexure is less than the curvature of any circle, or

FIG. 164. n. 1. is of that kind which was described in cor. 11. For when E is at this confine, the tangent ET meets the curve in the point E only; the cube of ET is to the solid contained by TM, T*m* and Tm in an invariable ratio: therefore the rectangle ETK is to *m*Tm in the same ratio; consequently EG is an asymptote of the curve FK in this case, and (by cor. 11.) the

FIG. 164. n. 2. curvature at E is less than that of any circle. It is the contrary when E is a cuspid in any line of this order: for the curvature at the cuspid is always greater than that of any circle, or is of the kind described in the last corollary; because, in this case, while T approaches to the cuspid E, the ratio of ET to T*m* may become greater than any given ratio; consequently, the ratio of Tm to TK may likewise become greater than any given ratio, and when m comes to *g*, K must come to E: therefore the curve FK passes through E, and the curvature at the cuspid E is greater than that of any circle, by the last corolla-

FIG. 163. ry. In these lines the variation of curvature is found by determining the tangent of the curve FKB at B when GET is a right angle;

angle; and it is easy to shew, that if GR and *gr* touch the curve at G and *g*, and meet EA in R and *r*, respectively; and if BV the tangent of FKB meet EA in V; then EV shall be to EA as the rectangle RE*r* is to the sum, or difference, of the rectangle RA*r* and the square of EA.

380. Cor. XIV. When EMH is a geometrical curve, BKF is likewise geometrical, since TM, TE and TK are in continued proportion; and since the point B is determined by the intersection of a geometrical curve with the right line EB, it follows, that, when the angle BET is given, the curve in which B is always found is likewise geometrical; and according as EMH is geometrical, or mechanical, the curve in which the center of curvature of EMH is always found is geometrical, or mechanical. By this proposition and the preceeding corollaries, the curvature of EMH and its variation, with the degree of contact of the curve and circle of curvature, may be determined when EMH is geometrical. We now proceed to the theorems by which these are determined with equal facility when it is mechanical. Fig. 149. & 150.

381. Let E*m* be a common parabola, ET the tangent at E, E*d* the diameter through E, EB the parameter of this diameter, and let any right line T*m* parallel to E*d* meet the tangent in T and the parabola in *m*. Let *ef* a right line given in position meet E*d* in *e* and T*m* in *f*; then, if *fn* be taken upon *fm* so that the triangle *efn* be always equal to the rectangle contained by T*m* and a given right line DG, the point *n* shall be always found in a right line *en* given in position. For, since the rectangle contained by T*m* and EB is always equal to the square of ET, (by the known property of the parabola,) the triangle *efn* is to the square of ET as DG is to EB. Because ET and *ef* are given in position, and *f*T is always parallel to *e*E, the square of ET is to the square of *ef* always in a given ratio; consequently, the triangle *efn* is to the square of *ef*, and *fn* is to *ef*, in a given ratio: Therefore *en* is a right line given in position. And, conversely, when ET, *ef* and *en* are given in position, if the rectangle contained by T*m* and a given right line DG be always equal to the triangle *efn*, the point *m* shall be in a parabola that touches ET in E so as to have its diameters Fig. 165.

 paral-

parallel to Tm, and the parameter of the diameter through E in the same ratio to DG as the square of ET is to the triangle esn; for supposing EB to be to DG in this ratio, the rectangle contained by Tm and EB shall be equal to the square of ET.

382. Let the base AP flow uniformly, and its fluxion be represented by any given right line DG; let PN the ordinate of the figure DeNP always measure the fluxion of PM the ordinate of DEMP as in prop. 20. let en the tangent of eN at E and es parallel to DP meet PN in n and s, and ET the tangent of EM meet it in T; then, when DP becomes equal to DG, sn shall measure the fluxion of the ordinate De, (by prop. 14.) or the second fluxion of DE; and ET shall measure the fluxion of the curve HE.

PROP. XXXIII.

Fig. 165. *The base* AD *being supposed to flow uniformly, let the second fluxion of the ordinate* DE *be to the fluxion of the curve* HE *as the fluxion of the curve is to* Eb; *and if* EB *be taken upon* DE *from* E *on the concave side of the curve equal to* 2Eb, *the circle of curvature at* E *shall pass through* B.

Let the rectangle contained by Tm and the given right line DG be always equal to the triangle esn, and, by the last article, the point m shall be always found in a parabola Em that touches ET in E, whose diameter through E is Ed; and the parameter of this diameter must be equal to EB, because it is to DG as the square of ET (or the rectangle contained by sn and Eb when DP becomes equal to DG) is to the triangle esn (or the rectangle contained by sn and one half of es, or DG;) that is, as EB is to DG. But no parabola can be described through E betwixt the curve EM and the parabola Em. For, let Eu be any other parabola described through E, touching ET, and having Ed for its diameter; let TM meet this parabola in u, and the rectangle contained by Tu and DG be always equal to the triangle esx; then the point x shall be in a right line ex given.

given in position, by the last article; if Eb be the parameter of the diameter of E*u* that passes through E, the triangle *esx* shall be to the square of ET as DG is to Eb, and (because the square of ET is to the triangle *esn* as EB is to DG) the triangle *esx* to *esn* as EB is to Eb, so that *sx* is to *sn* in the same ratio. The rectangle contained by TM and DG is always equal to the trilineal area *es*N, by what was shewn in prop. 20. Therefore the right lines TM, T*m* and T*u* are in the same proportion to each other as the area *es*N, the triangle *esn*, and the triangle *esx*. But, because *en* is the tangent of the curve *e*N, if we suppose PM to move towards DE, the area *es*N and triangle *esn* shall either become both greater or both less than the triangle *esx* (by art. 181.) according as *sn* is greater or less than *sx*, or Eb is greater or less than EB; consequently, TM and T*m* shall either become both greater or both less than T*u* while M approaches to E; and therefore the parabola E*u* cannot pass between the curve EM and parabola E*m*. In the same manner, no other parabola can pass between EM and E*m*; but an indefinite number of parabolas can be described between EM and any other parabola E*u*, as an indefinite number of right lines may be drawn through *e* betwixt *en* and *ex* within the angle *nex*. Therefore the curve EM and the parabola E*m* have the same circle of curvature, by the latter part of art. 373. and this circle is that which is described upon EB so as to touch the right line ET, by the same article.

383. Cor. I. As all curves that pass through E have the same tangent when the first fluxions of their ordinates are equal, the fluxion of the base being given; so they have the same circle of curvature at E when the second fluxions of their ordinates are likewise equal. Thus, the motion with which the base flows being given, the position of the tangent depends on the motion with which the ordinate flows, and the flexure of the circle of curvature depends on the acceleration or retardation of this motion. When the angle BET and the fluxion of the base are given, the ray of curvature EC is reciprocally as the second fluxion of the ordinate; because EC is to E*b* (which is reciprocally as *sn*) as ET is to DG. When the fluxion of the base is constant, the solid contained by EC the ray

of curvature, DG which measures the fluxion of the base, and *fn* that measures the second fluxion of the ordinate, is always equal to the cube of ET which measures the fluxion of the curve.

Fig. 167. If the curve ER be generated from *er* in the same manner as EM is generated from *e*N, (that is, if the rectangle contained by DG and TR be always equal to the area *efr*,) the contact of EM and ER shall be always one degree closer than that of *e*N and *er*. Thus, if *e*N and *er* pass through the same point *e*, EM and ER shall have the same tangent; when they have the same tangent, EM and ER have the same curvature; when *e*N and *er* have the same curvature, the contact of EM and ER with their common circle of curvature is of the same kind, or they have the same variation of curvature, and so on.

Fig. 166. 384. Cor. II. The second fluxion of the curve HE is to the first fluxion of the curve as the fluxion of the ordinate DE is to E*b*: And, S being any given point in the line D*d*, if SK be always perpendicular to the tangent of HE in K, the fluxion of the perpendicular SK shall be to the fluxion of SE as SK is to E*b*, or as SE is to the ray of curvature EC. For, let any ordinate pm meet the curve EM in m and *e*N in n; let nf parallel to the base meet GM in f, and S*k* be perpendicular to the tangent at m in *k*: then D*f* and Df shall be parallel to the tangents of the curve HEm at E and m, respectively, and shall measure the fluxions of the arches HE and Hm, the fluxion of the base being measured by DG, or *ef*. Let *ny* be perpendicular to D*f* in *y*; and, since the fluxion of D*e* is measured by *fn*, the fluxion of D*f* (or the second fluxion of the curve HE) shall be measured by *fy*, by art. 193. But *fy* is to D*e* as *fn* is to D*f*, or (by this proposition) as D*f* is to E*b*. Therefore the second fluxion of the curve (which is measured by *fy*) is to the fluxion of the curve (or D*f*) as the fluxion of the ordinate (or D*e*) is to E*b*. This likewise appears from art. 96. because the fluxion of the square of ET, or D*f*, is equal to the fluxion of the square of D*e*, DG being invariable. Let the angle SK*x* be made equal to SET or D*f*G, and K*x* shall be the tangent of the curve K*k* that passes always through the intersections of the tangents of HEM and the perpendiculars from S, by art. 211. The angle KS*k* is equal to *f*Df, and the angular velocity of SK about S is equal to the

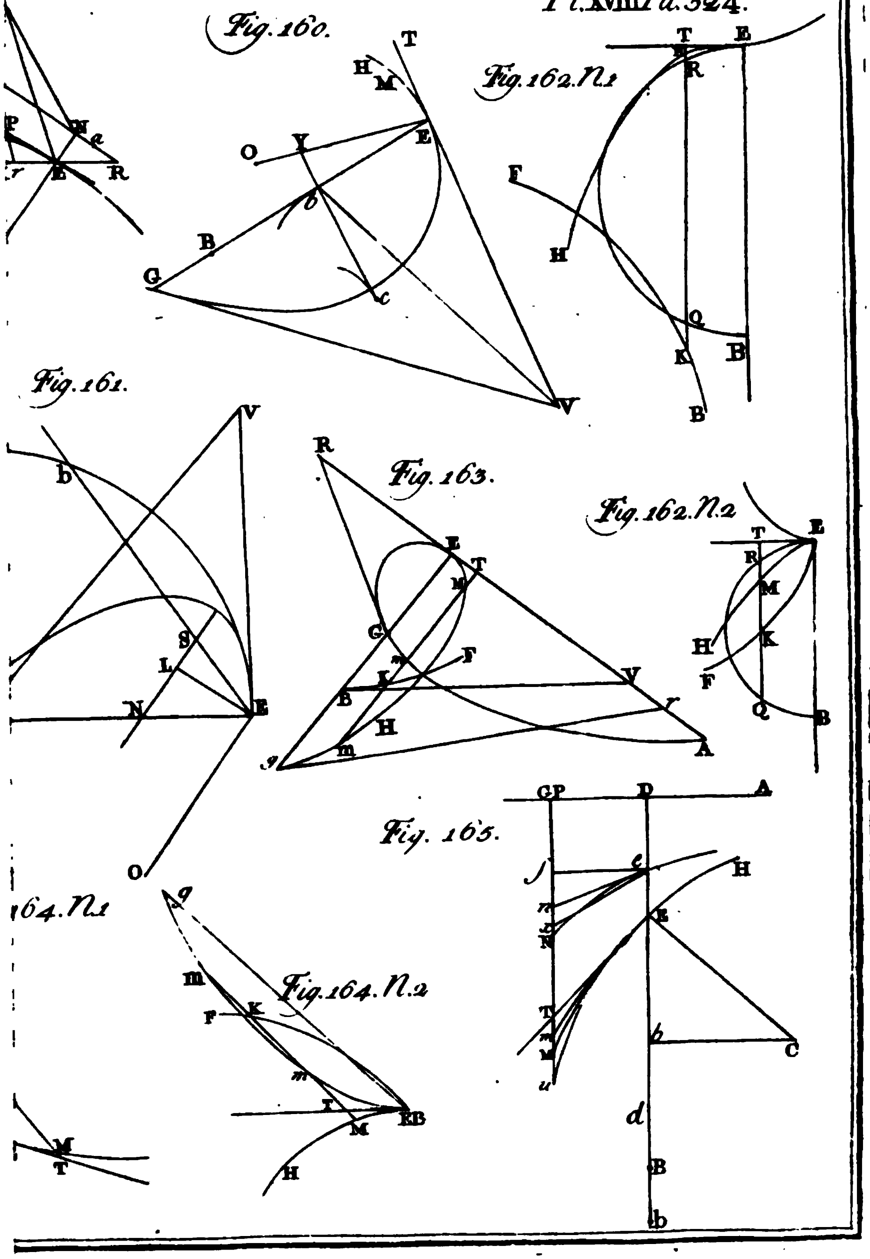


Fig. 160.
Fig. 161.
Fig. 162. N. 1
Fig. 162. N. 2
Fig. 163.
164. N. 1
Fig. 164. N. 2
Fig. 165.

the angular velocity of D*ſ* about D. Therefore the fluxion of SK is to the fluxion of D*ſ* as SK is to D*ſ*; and the fluxion of SK is to SK as the fluxion of D*ſ* is to D*ſ*, that is, (by what has been ſhewn,) as the fluxion of DE is to E*b*. But the fluxion of SE is equal to the fluxion of DE. Therefore the fluxion of SK is to the fluxion of SE as SK is to E*b*, or as SE is to EC.

385. Cor. III. Let S be a given point in the line D*d* as in the laſt article, EV and EM any lines through E that meet *p*M parallel to SD in V and M; let S*p* parallel to DP meet PM in *p*, join SV and SM; and the difference of the ſecond fluxions of SM and *p*M ſhall be equal to the difference of the ſecond fluxions of SV and *p*V when M and V come to E. For, the difference of the ſquares of SV and *p*V is equal to the difference of the ſquares of SM and *p*M: from which it follows, (by art. 96. & 99.) that, ſince the firſt fluxions of SV and *p*V become equal, and the firſt fluxions of SM and *p*M become likewiſe equal, when V and M come to E, and SV, *p*V, SM, *p*M then coincide with each other, the difference of the ſecond fluxions of SM and *p*M is to the difference of the ſecond fluxions of SV and *p*V at that term, as the ſum of SV and *p*V is to the ſum of SM and *p*M, and therefore in a ratio of equality. When the baſe flows uniformly, and EV is a right line, the ſecond fluxion of PV vaniſhes; therefore the ſecond fluxion of SV is in this caſe the ſame when V ſets out from E in any right line that does not coincide with ES, and is equal to the difference of the ſecond fluxions of SM and *p*M at the ſame term, whatever curve be deſcribed by the point M. When EV is a circle that has its center in S, the ſecond fluxion of the ordinate PV is equal to the ſame difference of the ſecond fluxions of SM and *p*M, when M comes to E, becauſe in this caſe SV has no fluxions of any order whatſoever: And if E*u* any right line through E given in poſition meet PV in *u*, the ſecond fluxionof S*u* ſhall be equal to the ſecond fluxion of PV, the ordinate from the circle, when V and *u* come to E. It appears likewiſe, that the ſecond fluxion of SM is the ſame when M comes to E in all lines that have the ſame curvature at E, and is meaſured by the difference betwixt *ſu* which repreſents the ſecond fluxion of DE, and a third proportional

tional to SE and DG (which represents the invariable fluxion of AD;) because the fluxion of the circular arch EV becomes equal to the fluxion of AD when V comes to E, and the second fluxion of PV the ordinate of the circle is to DG as DG is to SE the radius of the circle, by this proposition.

FIG. 167. 386. COR. IV. The rest remaining as in art. 384. let EB be perpendicular to the tangent ET and to AD, and, since *sy* which measures the fluxion of ET (or the second fluxion of the curve HE) vanishes, the fluxion of the rectangle contained by *sn* and EC (which in this case coincides with E*b*) shall likewise vanish, because that rectangle is equal to the square of ET; and the fluxion of EC the ray of curvature is to the fluxion of *sn* as EC is to *sn*, by prop. 3. Let *er* be a parabola that has the same curvature with *e*N at *e*, and the rectangle contained by TR and DG be always equal to the area *esr*, as in art. 383. Let the rectangles MTK, *m*TQ, RT*k* be each equal to the square of ET, as in art. 366. and let T*k* meet B*t* the tangent of B*k* (or of BK) in *t*. Then, the fluxion of the base being constant, the fluxion of *sn* shall be represented by 2*nr*, or 6*m*R, by art. 255. But, since T*m* is to TR as T*k* is to TQ, 6*m*R (which measures the fluxion of *sn*) is to 6*k*Q as T*m* is to T*k*; and therefore, as the rectangle *m*TQ is to the rectangle contained by T*k* and TQ, or EB. The rectangle *m*TQ is equal to the square of ET, or (by this proposition) to the rectangle contained by *sn* and E*b*; consequently, the fluxion of *sn* is to 6*k*Q as *sn* is to 2T*k*, and the fluxion of the ray of curvature EC is to 6*k*Q as EC is to 2T*k*, or as EB is to 4T*k*: from which it follows, that the fluxion of the ray of curvature is measured by three halves of Q*t*, the fluxion of the curve being measured by ET or BQ; and that the variation of curvature, according to Sir ISAAC NEWTON, (who measures it by the ratio of the fluxion of the ray of curvature to the fluxion of the curve,) is measured by three halves of the tangent of the angle QB*t* in which the curve BK intersects the circle of curvature, as was observed in art. 369. But, since the curvature itself is reciprocally as the ray of the circle of curvature, if we should therefore measure its variation by the ratio of its fluxion to the fluxion of the curve, this variation would be as the tangent of the angle QB*t* direct-

directly and the square of the ray of curvature inversely, and would be as the measure of the angle of contact contained by the curve and circle of curvature; because, when a quantity A is always inversely as another quantity B, its fluxion is as the fluxion of B directly and the square of B inversely. However, to avoid confusion, we have conformed to Sir ISAAC NEWTON's explication of the variation of curvature.

387. COR. V. When *en* the tangent of *e*N is parallel to the base, and the rectangles contained by DG and TM, and by DG and T*m*, are respectively equal to the areas DPN*e*, DP*ne*, as in prop. 20. then E*m* becomes a right line, and the curvature of EM is of the kind that is less than the curvature of any circle, which was described in art. 377. In this case, the second fluxion of the ordinate DE vanishes; and (provided the curve be continued on both sides of DE) if the number of the fluxions of DE of successive orders that vanish be an odd number, then E is a point of contrary flexure; but if it is an even number, E is not a point of that kind, as was shewn in art. 266. FIG. 165.

388. COR. VI. The same things being supposed as in art. 384. it follows from art. 202. & 208. that the velocity of the angular motion of the tangent of HEm (or of SK about S) when m sets out from E, is the same when the point n describes any curve that has the same tangent at *e*, or when m describes any line that has the same tangent and curvature at E, the fluxion of AD being given. Therefore the angular velocity of the tangent of EM at E is equal to the angular velocity of the tangent of the circle of curvature, or of the ray of curvature about the center C, the fluxion of AD being given, because the angular velocity of the tangent of a circle is equal to the angular velocity of the ray drawn from the center to the point of contact, by art. 18. In like manner, if B be any point in the circle of curvature, and the right lines EM, BM revolve about E and B so that their intersection M describe the curve HME, the angular velocities of EM and BM shall be equal when M comes to E. For, suppose the arch ER of the circle of curvature to be within EM the arch of the curve EMH; and, if the angular velocity of EM about E be said to be less than the angular velocity of BM about B at the term when M comes to FIG. 166. FIG. 168.

E,

E, let it be less in the ratio of EB to EZ, and (by art. 209.) it shall be equal to the angular velocity of ZM about Z at that term, and therefore less than the velocity of *b*M about *b* at the same term, *b* being any point betwixt B and Z. Upon E*b* describe a circle E*rb* that touches ET, produce EM till it meet this circle in *r*, join *br* and *b*M; and, since the arch EM is within this circle E*rb*, (art. 364.) the angle MET, or E*br*, is always greater than E*b*M while M describes EM: therefore the angular velocity of EM about E cannot be less than the angular velocity of *b*M about *b* when M comes to E; but it was supposed equal to the angular velocity of ZM about Z at that term; and these are contradictory. In the same manner, it is shewn, that the angular velocity of EM about E is not greater than the angular velocity of BM about B when M comes to E, and that these velocities are equal likewise when EM is within the circle of curvature ER.

389. The various properties of the circle suggest various methods of determining the circle of curvature. The two following propositions are deduced from the properties of this circle that were described in the last article, and sometimes give more simple contructions for determining the ray of curvature than the preceeding propositions.

PROP. XXXIV.

Let S *be any given point in the plane of the curve* BL, *and* SP *be always perpendicular from it on* LP *the tangent of* BL; *let* SV *perpendicular to* SL *meet* LC *the ray of curvature in* V: *and* LC *shall be to* LV *as the angular velocity of* SL *about* S *is to the angular velocity of* SP.

FIG. 169.

Let CI perpendicular to SL meet it in I, and, since the angle LCI is equal to SLP, it follows from what was shewn in prop. 18. that when L describes the curve BL, the angular velocity of CL about C is equal to the angular velocity of IL about I, which is to the angular velocity of SL about S as SL is to IL,

IL, (by art. 209.) or as LV to LC. But the angular velocity of CL about C is equal to the angular velocity of SP about S, by art. 388. Therefore LC is to LV as the angular velocity of SL about S is to the angular velocity of SP about S, or, SA being a right line given in position, as the fluxion of the angle ASL is to the fluxion of the angle ASP. This may be demonstrated likewise from art. 384. where it was shewn, that LI is to SP as the fluxion of SL is to the fluxion of SP; from which it follows, that LI is to LS (or LC to LV) in the ratio compounded of that of the fluxion of SL to the fluxion of SP and that of SP to SL; that is, (by art. 202. 208. 211. & 212.) as the fluxion of the angle ASL to the fluxion of ASP.

390. Cor. I. Let LI be bisected in Q, and PM be the ray of the circle of the same curvature with DP (the line in which P is always found) at P; then shall SQ, SL and 2PM be in continued proportion. For, let PT be the tangent of DP at P and ST be always perpendicular to PT, let PM meet LC in K; then, because the angles SPT, SLP are equal, (art. 208. & 209.) and KPT, KLP are right, the angles KPS, KLS are equal, a circle passes through the four points K, L, P, S, and the angle KSP is right. Therefore, by this proposition, MP is to KP, or SL, as the angular velocity of SP is to the angular velocity of ST, or as the fluxion of the angle ASP to the fluxion of 2ASP — ASL, because the angles PST, LSP are always equal. But SL is to LI (or 2LQ) as the fluxion of ASP is to the fluxion of ASL. Therefore PM is to SL as SL is to 2SQ, or SQ, SL and 2PM are in continued proportion.

391. Cor. II. When the angular velocity of SL is to the angular velocity of SP in any invariable ratio, LC is to LV, or LI to LS, in the same invariable ratio. Thus, when AL is a common parabola, and A is the vertex, S the focus, the angle ASL is double of ASP and LI is double of LS; which agrees with what was shewn in art. 371. When AL is the logarith- Fig. 170.
mic spiral, and S is the center of the spiral, the angular velocity of SP is equal to the angular velocity of SL, because the angle LSP is invariable; therefore, in this figure, C coincides with V, or I with S; and a perpendicular to SL at S intersects LC perpendicular to the curve in the center of curvature. When

FIG. 169. AL is an equilateral hyperbola, the angular velocity of the tangent at L, or of the perpendicular SP, is equal to the angular velocity of SL, (by what was shewn at the end of art. 315.) and LC is equal to LV, or LI to LS; but they are on opposite sides of the point L. When the angular velocity of SL is to the angular velocity of SP in an invariable ratio, the angular velocities of SP and ST are likewise in an invariable ratio; and PM is to PK, or SL, in the same ratio. For example, when AL is a circle, and S is in the circumference, PM is two thirds of SL. In this case P is in an epicycloid that is described by a point in the circumference of a circle while it revolves on an equal circle; and M is in an epicycloid of the same kind.

FIG. 171. 392. COR. III. Let AE be a right line given in position, SA a perpendicular to it in A from the given point S; and let any other right line from S meet it in M; let the angle ASL be always to the angle ASM in any invariable ratio expressed by that of n to unit; and, SA, SM being the two first terms of a geometrical progression, let SL be equal to the term of this progression whose place in the series is denoted by $n+1$; or, more generally, let SL be to SA as the power of SM, whose exponent is any positive number n, is to the same power of SA: then the angle SLP (contained by SL and the tangent at L) shall be equal to the angle SMA; the ray of curvature LC shall be to LV as n is to $n-1$; and, if SB be to SA as $n-1$ is to $n+1$, the variation of curvature at L shall be as AM directly and as SB inversely. For, let circles described from the center S through M and L meet SA in F and f; and, the points F, f remaining fixed while M and L are supposed to proceed in the lines AM and AL, the fluxion of the arch FM shall be to the fluxion of fL in the ratio compounded of that of SM to SL and that of the fluxion of the angle ASM to the fluxion of ASL, (or of unit to n;) the fluxion of SM is to the fluxion of SL in the same ratio, by art. 167. consequently, the fluxion of SL is to the fluxion of SM as the fluxion of the arch fL is to the fluxion of FM. Therefore the angle SLP is equal to SMA, by prop. 16. and if SP be perpendicular to LP, the angle ASL shall be to ASP in the invariable ratio of n to $n-1$. The fluxion of the angle ASL is to the fluxion of ASP, and

(by

(by this propoſition) the ray of curvature LC is to LV, in the ſame ratio. The fluxion of the ray of curvature LC is to the fluxion of LV as n is to $n-1$, (art. 24.) the fluxion of LV is to the fluxion of SL in the ratio compounded of that of $n+1$ to n and that of LV to SL, (becauſe LV is to SL as SM is to SA, or as the power of SL whoſe exponent is $1+\frac{1}{n}$ is to the ſame power of SA;) and the fluxion of SL is to the fluxion of the curve AL as LP is to SL, or AM to SM. Therefore the fluxion of the ray of curvature LC is to the fluxion of the curve AL as AM is to SB; and the variation of curvature (as it is underſtood by Sir ISAAC NEWTON, and was explained in art. 369.) is as AM directly and SB inverſely. Hence, the variation of curvature at L in any given figure of this kind is as the tangent of the angle contained by SL and the perpendicular to the curve, and in any of thoſe figures is to the variation at the point in the parabola where the right line from the focus interſects the curve in an angle equal to SLP, as SA is to 3SB. In the logarithmic ſpiral AL the fluxion of the ray of curvature FIG. 170. is to the fluxion of the curve as SC is to SL, or LP is to SP; and, conſequently, in the invariable ratio of the tangent of the angle LSP to the radius. Therefore the variation of curvature in this figure is invariable, (as appears likewiſe from art. 370.) and is to the variation of curvature at the point of a parabola where the right line from the focus interſects the curve in an angle equal to the given angle SLP as 1 is to 3.

393. COR. IV. If we ſubſtitute a ſemicircle AMS in the place FIG. 172. of the right line AE in the conſtruction of the laſt article, it will appear in the ſame manner that the angle LSP is equal to ASM, (SP being now the ſame way from SL that SM is from SA,) that the ray of curvature LC is to LV as n is to $n+1$, and that (SM being produced till it meet Am perpendicular to SA in m and SB being taken to SA as $n+1$ is to $n-1$) the variation of curvature at any point L is as Am directly and SB inverſely, and in any given figure of this kind is as AM or the tangent contained by SL and the perpendicular to the curve.

394. COR. V. If any curve AM be ſubſtituted for the right FIG. 173. line AM in cor. 3. the reſt of the conſtruction remaining, it will

appear in like manner that the angle SLP contained by SL and LP the tangent of AL shall be equal to the angle SMT contained by SM and the tangent of AM. Let M*c*, LC be the rays of curvature at M and L, respectively; let S*u*, SV perpendicular to SM and SL meet those rays in *u* and V; and let *cr* be to *cu* as unit is to *n*: then LC shall be to LV as M*c* is to M*r*. The demonstration is easily deduced from the equality of the angles SLP, SMT, by this proposition. Thus, if AM be a conic section, S the center, SA half the transverse axis, the angle ASL equal to 2ASM, and SA, SM, SL be in continued proportion; then AL shall be a conic section that shall have its focus in S, and, *cu* being bisected in *r*, LC shall be to LV as M*c* is to M*r*.

FIG. 123. 395. COR. VI. Let CP and SP revolve about the poles C and S with any angular velocities that are to each other in the invariable ratio of ST to CT, as in art. 315. & 316. make the angle CPN equal to SPT and TP*l* equal to TPS, and let P*l* meet CS in L; let PO perpendicular to PN meet CV perpendicular to CP in V, and let PG be to PV in the invariable ratio of ST to CT: then, if PO be the ray of curvature of the line described by P, PO shall be to PG as TL is to the sum or difference of TL and ST according to the different positions of the points T and L with respect to the poles C and S. The demonstration is deduced from prop. 18. and art. 315. by this proposition. If SMC be an arch of a circle, the arch SZ be to SM in any given ratio, and CZ always meet the right line SM in P, the tangent of the line described by the point P is determined by art. 315. and its curvature by this construction; because the angular velocity of P is to the angular velocity of CM, or of SP, in the same given ratio.

396. The following proposition is the sequel of the 18th and 26th, and is of use in enquiries concerning the curvature of lines that are described by means of right lines revolving about given poles, or of angles that either revolve about such poles or are carried along fixed lines, in the manner explained by several examples in the preceeding chapter from art. 315. to art. 324.

PROP.

Fig. 166. N. 2.

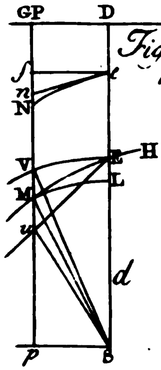


Fig. 167.

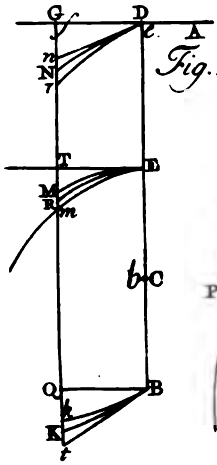


Fig. 169. N. 1.

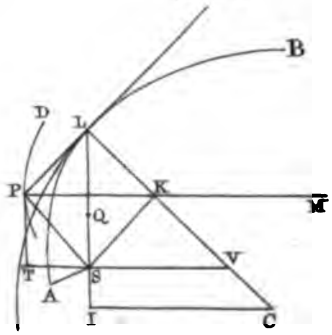


Fig. 168.

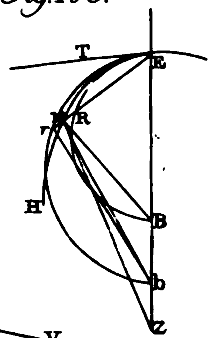
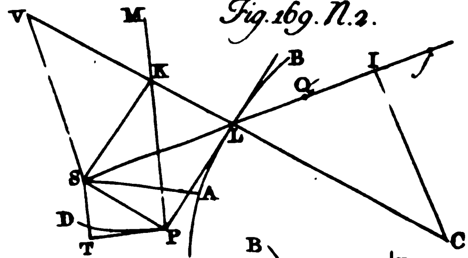


Fig. 169. N. 2.



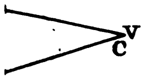
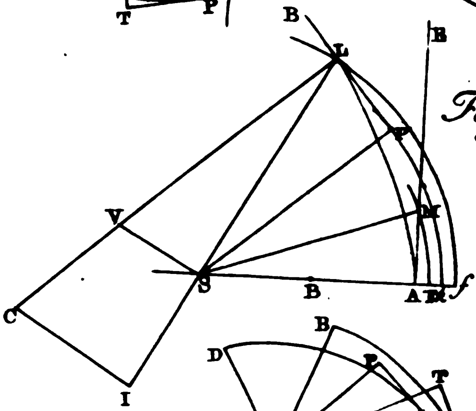


Fig. 171.



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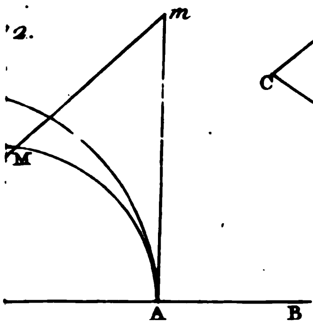
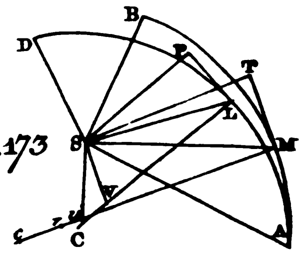


Fig. 173.



PROP. XXXV.

Let S *be a given point in the plane of the curve* HEM, *join* EM *and* SM, *and when the point* M, *in describing the curve* MEH, *comes to* E, *let the angular velocity of* SM *about* S *be to the angular velocity of* EM *about* E *as* ET *is to* ST, *and the point* T *be taken upon* SE *produced, or between* S *and* E, *according as the angular motions of* SM *and* EM *at that term have the same or contrary directions; then, if* ST, SE *and* SB *be in continued proportion, and* B *be taken upon* SE *on the same side of the point* S *with* T, *the circle of curvature at* E *shall pass through* B. FIG. 174.

For, since SB is to SE as SE is to ST, BE is to SE as ET is to ST, or as the angular velocity of SM about S to the angular velocity of EM about E at the term when M comes to E, by the supposition. Join BM, and the angular velocity of SM about S shall be to the angular velocity of BM about B, when M comes to E, as BE is to SE, by art. 209. Therefore the angular velocity of EM about E is equal to the angular velocity of BM about B at that term; and, by art. 388. the circle of the same curvature with EM at E passes through B. The concavity or convexity of the arch EM is towards S, according as the angular motions of SM and BM have the same or contrary directions while M comes to E in describing ME.

397. COR. I. Let the invariable angles DEG, KSH revolve about the poles E and S after the same manner as in art. 319. Let N the intersection of SK and ED move in any line *Ff*, and M the intersection of SH and EG describe the curve EM. When SH coincides with SE, let N come to *n*, EG to E*g*, and let the right line A*n* touch *Ff* in *n*; make the angle E*n*T equal to S*n*A the contrary way from E*n* that S*n*A is from S*n*; take SB from S towards T upon ST, a third proportional to ST and SE, and a circle described upon the chord EB so as to FIG. 175.

touch

touch E*g* ſhall be the circle of the ſame curvature with EM at E. The curvature at S is determined in the ſame manner; and, if SK touch the curve F*f* at *u* when EG coincides with ES, then S ſhall be a point of contrary flexure in the curve EMS; and, if F*f* has a continued curvature at *u*, the curvature of EMS at S ſhall be leſs than that of any circle. When the right line S*n* is itſelf the tangent of F*f* at *n*, and F*f* has a continued curvature at *n*, the point E is a cuſpid, and the curvature at E is greater than in any circle.

FIG. 176. 398. COR. II. Suppoſe that it is required to deſcribe a conic ſection through the points S and M, that ſhall touch a given right line E*g* at E ſo that the circle of curvature at E ſhall meet SE in B. Let SA be taken from S the ſame way with SB, ſo that SB, SE and SA may be in continued proportion; let the angle SE*g* revolve about E; and, when the ſide E*g* comes to EM, and ES to E*d*, let SM interſect E*d* in N, join AN; let the right line MN revolve about the point S, and its interſection with the ſide EN move over the right line AN, then ſhall its interſection with the other ſide EM deſcribe the conic ſection required.

FIG. 177. 399. COR. III. In the ſame manner, if it is required to deſcribe a line of the third order through the double point S, the three points M, C and K ſo as to touch E*g* in E, and EB the chord of the circle of curvature at E be given; let the point A be determined as in the laſt problem; let the angle SE*g* revolve about E and a right line MN about S, in the ſame manner; find three points N, *c* and *k* from M, C and K as N was found from M in the laſt problem, deſcribe a conic ſection through S, A, N, *c* and *k*; and, if the interſection N move always in this conic ſection, M ſhall deſcribe the line of the third order required. If it be required that the line ſhall have a point of contrary flexure at E, the conic ſection is to be deſcribed through N, *c* and *k* ſo as to touch SE at S.

FIG. 136. 400. COR. IV. The five points A, B, C, S and E in a conic ſection being given, let it be required to determine the circle of curvature at C. Determine the points D, *q*, *n* and the tangent C*n* as in art. 324. make the angle D*qr* equal to S*q*A the contrary way from D*q* that *q*A is from S*q*, and let *qr* meet SD in *r*; make the angle D*nx* equal to C*n*A with the like precaution,

tion, and let nx meet CD in x; join rx, and let it meet CS in T; let ST, SC and SB be in continued proportion, and the points B and T be on the ſame ſide of S: then a circle deſcribed upon the chord CB ſo as to touch Cn ſhall be the circle of curvature at C. There ariſe various conſtructions for determining the curvature of lines of the higher orders from this propoſition, analogous to thoſe by which the tangents and aſymptotes of lines were determined in the laſt chapter. But, inſtead of inſiſting on theſe, it will be more worth while to add here a property of the lines of the third order to thoſe that have been obſerved by Sir ISAAC NEWTON, *Enumer. linear. tertii ordinis.*

401. Let A be any point in a line of the third order from which two tangents AC, AS can be drawn to the curve in C and S; from any point in the curve, as P, draw right lines to C and S that meet the curve again in M and N, reſpectively; join CN and SM, and the interſection of theſe lines ſhall be always in the curve. If CN and SM be parallel to each other, they ſhall be parallel to the aſymptote of an hyperbolic branch, or ſhall ſhew the poſition to which the tangent of a parabolic branch continually approaches while the figure is produced; that is, according to the uſual ſtile, they ſhall interſect each other in the curve either at a finite or at an infinite diſtance: And if two other tangents Ac, A$ſ$ can be drawn from the ſame point A to the curve in c and $ſ$, the right lines that join any of the points of contact C, S, c and $ſ$ ſhall interſect each other in the curve. We may explain this property, with its conſequences, more fully on another occaſion, and ſhew how a line of the third order (whether it have a *double* point, or not) can be deſcribed through ſeven points ſo as to touch two right lines given in poſition at two of thoſe points; and ſhall only obſerve further, to illuſtrate this property, that it holds of any conic ſection and right line deſcribed in the ſame plane: for if CA and SA touch a conic ſection in C and S, and from any point P the right lines PC, PS be drawn that meet the conic ſection in M and N; then CN and SM ſhall always interſect each other in ſome point of the right line AP. Of this ſee Mr. SIMSON's *Sect. conic. lib.* 5. *prop.* 45. But to return to the curvature of lines: FIG. 178. FIG. 179.

402. A.

FIG. 180. 402. A flexible line or thread IC*a*A being applied along the convexity of the curve IC*a* from I to *a*, let the part *a*A be extended in a right line that touches the curve in *a*; then, while one extremity of the line, or thread, remains fixed at I, let the other extremity A move towards H, so that the line may be gradually separated from the curve, and the part CE which is not applied to it be always extended in a right line that touches the curve; then the point E shall in this manner trace a curve AEM that by the excellent Mr. HUYGENS is said to be described by the evolution of *a*CI, which is itself called the *evoluta*.

PROP. XXXVI.

Let the right line CE *touch the evoluta* aCI *in* C, *and meet the curve* AEM, *that is described by the evolution of* aCI, *in* E; *then a circle* ERB *described from the center* C *through the point* E *shall have the same curvature with* AEM *at* E: *and if* c *be the center of the circle of the same curvature with the evoluta* aCI *at* C, *the variation of the curvature of* AEH *at* E *shall be measured by the tangent of the angle* CEc.

Let the right line QM touch the evoluta at any point Q betwixt C and I, and meet the curve AEH, the circle ERB and right line EB in M, R and N; join CR, and, since the right line QM is equal to the arch QC and right line CE (or CR) taken together, QM is greater than QR; consequently, the arch EM of the curve EMH passes without the circle ERB, and it is manifest that no circle described through E with a radius less than CE can pass between EM and ER. Because the sum of QN and NC is greater than the arch QC, (art. 183.) the sum of QN and NE is greater than the sum of the arch QC and ray CE, or QM; therefore NE is greater than NM: and a circle described from any point N through E, with a radius NE greater than CE, passes without EM and ER. Therefore no circle can pass between EM and ER, and ER is the circle of

of the ſame curvature with EM at E, by art. 364. In like manner it is ſhewn, that the circle RE produced on the other ſide of E paſſes without EA, and that no circle can be drawn between them on that ſide. It is evident, that the curvature decreaſes from E towards H, but increaſes from E towards A; and that the curve paſſes within or without the circle of curvature, according as the curvature increaſes or decreaſes. The variation of curvature at E is meaſured by the ratio of the fluxion of the ray of curvature CE (or of the curve *a*C) to the fluxion of the curve AE, or (becauſe the angular motion of CE the tangent of *a*CI is equal to the angular motion of *c*C about *c*, by art. 388.) by the ratio of C*c* to EC, that is, by the tangent of the angle *c*EC.

403. Cor. I. The length of the arch CQ is equal to the difference of the rays of curvature MQ and EC, when the curvature increaſes or decreaſes continually from E to M. Hence, from any geometrical curve another may be deduced that ſhall admit of an accurate rectification. If we ſuppoſe, (as in art. 366.) that TMK is always parallel to EC, the rectangle MTK equal to the ſquare of ET, and BV the tangent of BKF at B to meet TE in V, then ſhall EV, EC and one third part of C*c* the ray of curvature of the evoluta *a*CI be in continued proportion. For, (by art. 386.) when K ſets out from B, the fluxion of TK is to the fluxion of the ray of curvature EC as 2 is to 3; conſequently, BE is to EV as 2C*c* is to 3EC: but BE is equal to 2EC, (by prop. 32.) therefore EV is to EC as EC is to one third part of C*c*; and if B*r* perpendicular to the curve BKF meet ET in *r*, C*c* ſhall be equal to three fourths of E*r*. Thus the curve BKF, by which the curvature of EMH and its variation were determined in prop. 32. and its corollaries, ſerves likewiſe for determining the curvature of the evoluta *a*CI.

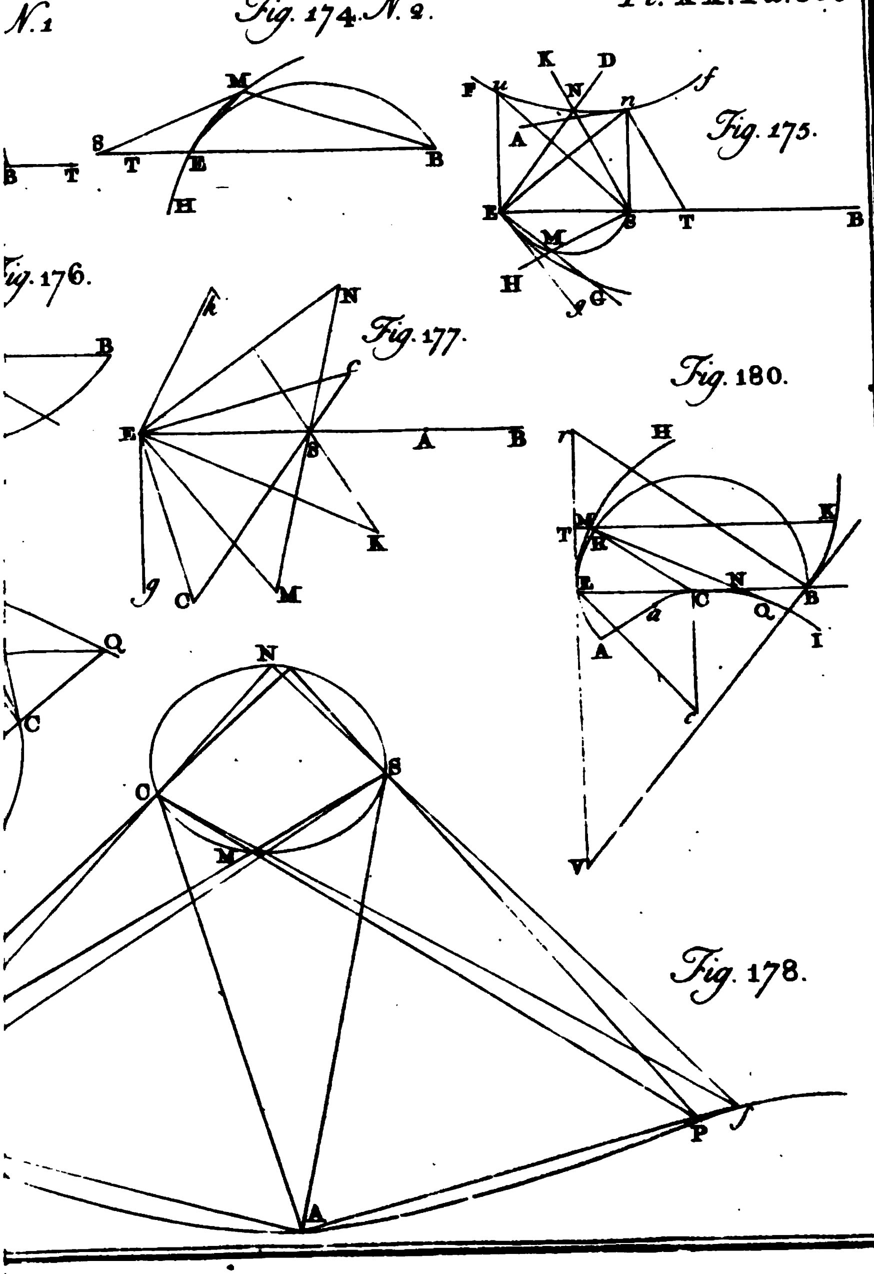
404. Cor. II. Hence a ready way is deduced for finding the center of curvature of the line by whoſe evolution any conic ſection is deſcribed. Let E be any point in a conic ſection, EO a diameter through E, *b* the center of curvature at E; let *b*Y parallel to the tangent ET meet EO in Y; take *bc* equal to 3*b*Y upon Y*b* produced from *b*, and *c* ſhall be the center of curvature of the line by whoſe evolution the conic ſection EMH is Fig. 160.

described. For the triangles VEb, EbY are similar, (by the 5th property of the circle of curvature, art. 375.) and VE is to Eb as Eb is to bY, and c is the center of curvature belonging to the point b of the evoluta, by the last corollary. When E is in the extremity of either axis, bY and consequently bc vanish, the evoluta has a cuspid where the curvature is such as
FIG. 171. was described in art. 378. In like manner, in art. 392. let CY parallel to the tangent LP meet LS in Y, let Cc be to CY as $n+1$ is to $n-1$, and c shall be the center of curvature of the line qCz by whose evolution LA may be described. And by a similar construction the center c is determined in art. 393.

405. But to proceed now to consider some of the useful pro-
FIG. 181. blems that have a dependence on the curvature of lines: When a circle moves upon a given right line so as always to touch this right line and apply the parts of its circumference successively to it, any given point in the circumference describes a *Cycloid*. Let IV be the given right line, and C the given point in the circumference of the circle KCL touch that right line first in I, and again in i after describing the curve ICAi while the circle makes a compleat revolution; bisect Ii in B, and let BA perpendicular to Ii meet the curve in A; then ICAi is the cycloid, Ii its base, AB its axis, and AMB the generating circle. If the circle proceed still in the same manner along the right line IV, the same given point of the circumference will describe an equal and similar figure in every revolution of the generating circle; from which it is obvious, that this figure is not of those that are called geometrical, which can never meet a right line in more than a certain definite number of points; for it may be continued till it meet the base Ii, or any right line parallel to Ii betwixt Ii and Ab, in any assignable number of points. It has however several remarkable properties, useful in philosophy; some of which we shall briefly demonstrate.

1. Any ordinate from the cycloid, as CP, perpendicular to the axis in P that meets the semicircle AMB in M is equal to the sum of the arch AM and of its right sine MP. For, let the generating circle KCL touch the base in K when the point that describes the cycloid comes to C; then, since the arch KC or BM is equal to the right line KI, and the semicircle BMA

to


N. 1
Fig. 174. N. 2.
Fig. 175.
Fig. 176.
Fig. 177.
Fig. 180.
Fig. 178.

to BI, (by the deſcription,) BK, or MC, is equal to the arch CL, or AM; and CP is equal to the ſum of the arch AM and its right ſine MP. 2. The tangent of the cycloid at C is parallel to the chord AM. For, let O be the center of the ſemicircle AMB, join OM; and, ſince the tangent of the circle at M is perpendicular to OM, it follows from the fourteenth propoſition, that the fluxion of the arch AM is to the fluxion of AP as OM is to PM, and the fluxion of PM to AP as OP is to PM; conſequently, the fluxion of CP (which is equal to the ſum of AM and PM) is to the fluxion of AP as BP is to PM, or PM to AP: therefore (by prop. 14.) the tangent of the cycloid at C is parallel to the chord AM. 3. Let CR parallel to BA meet A*b* parallel to BI in R; and, ſince the fluxion of AR (or PC) is to the fluxion of AP as PM is to AP, or CR, the fluxion of the area ACR is equal to the fluxion of the area A*x*MP, by art. 111. Therefore theſe areas are always equal, and the area ACI*b* is equal to the ſemicircle A*x*MBA. The right line BI is equal to the ſemicircumference AMB; conſequently, the parallelogram B*b* is quadruple of the ſemicircle, and the area ACIB is triple of it. 4. The arch of the cycloid AC is double of AM the chord of the arch A*x*M. For the fluxion of the arch AC is to the fluxion of AP as AM is to AP, by prop. 14. and the ſecond property. The right lines AB, AM, AP are in continued proportion; conſequently, (by art. 96. or 142.) the fluxion of the chord AM is to the fluxion of AP as AM is to 2AP. Therefore the fluxion of the curve AC is to the fluxion of the chord AM as 2AP is to AP, or as 2 is to 1; and the arch AC is to the chord AM in the ſame ratio, by art. 33. ſo that the ſemicycloid ACI is double of AB the diameter of the generating circle.

406. Hence, 5. the curve AE*a* deſcribed by the evolution of the ſemicycloid ACI is an equal ſemicycloid. For, upon I*b* produced from *b* take *ba* equal to BA, deſcribe the ſemicircle *bma*, let *bm* parallel to the chord AM meet this ſemicircle in *m*, and CE the tangent at C meet A*b* in L; then, becauſe the angle *b*AM is equal to A*bm*, the arch *bm* is equal to the arch AM, and the chord *bm* equal to the chord AM, which is equal and parallel to CL or LE; conſequently, E*m* is equal and paral-

parallel to bL or KI, which is equal to the arch BM or *am*; and Ep the ordinate from E perpendicular to *ba* is equal to the sum of the arch *am* and of its right sine *mp*. Therefore, by the converse of the first property, AEa is a semicycloid that has *ba* for its axis and *bma* for its generating semicircle. Hence the ray of curvature of the cycloid AEa at E is equal to $2bm$; and the variation of curvature is measured by the tangent of the angle *abm*, or of the angle contained by the tangent at E and the ordinate Ep, by art. 402. It appears likewise, that a heavy body may be made to describe any arch of a cycloid Ea by suspending it from a flexible line or thread ICE equal to $2ab$, that has one end fixed at I and is applied to the convexity of the semicycloid ICA from I to C, Ib being perpendicular to the horizon: for Ea will be described by the evolution of IC; and, by an equal and similar semicycloid touching Ib on the other side at I, a pendulum may be made to oscillate in this figure.

407. If the motion of the generating circle KCL upon the base IB, or of the point M in the semicircle BMA, be uniform, and its velocity be the half of that which would be generated by an uniform gravity by falling through the diameter BA, then, the velocity of C in describing the semicycloid ICA will be equal to that which would be acquired by the same gravity by falling through BP, or by falling along the arch of the cycloid IC from I, BA being supposed perpendicular to the horizon. For, since the arch AC is double of the chord AM, the fluxion of the arch AM is to half the fluxion of the arch AC (or the velocity of M in the semicircle is to half the velocity of C in the semicycloid): as AB is to BM, (by prop. 17.) or in the subduplicate ratio of AB to BP; and the velocities acquired by an uniform gravity by falling through AB and BP are in the same ratio, by art. 95. Hence, the time in which a heavy body describes a semicycloid ICA, or AEa, by descending along it by its gravity, is to the time in which it would fall through BA the diameter of the generating circle by the same gravity as the semicircumference of a circle is to its diameter. For that time is the same in which the point K describes IB, or the point M describes the semicircumference BMA, by an uniform motion:

tion with half the velocity that would be acquired by falling through BA; and the time in which a body falls from B to A by the same gravity is equal to the time in which it would describe BA with the same uniform motion, by art. 95.

408. In general, 7. let a body descend by its gravity from any point in the cycloid AEa, as H, along the curve HEa, the axis ba being perpendicular to the horizon; let the ordinate from H meet the axis ba in d; upon the diameter da describe the semicircle dna; and let Ep the ordinate from the body at any point E always meet this semicircle in n: then shall the motion of the point n in the semicircle anb be uniform so as to measure the time, and its velocity shall be equal to the half of that which would be acquired by falling through the right line fa a third proportional to ba and da. For the chord an is always to the chord am in the subduplicate ratio of da to ba, and the fluxion of the chord an to the fluxion of the chord am (or one half of the fluxion of aE by the fourth property) in the same ratio. The fluxion of the arch an is to the fluxion of the chord an (by prop. 16.) as ad is to dn, or in the subduplicate ratio of ad to dp. Therefore the fluxion of the arch an is to half the fluxion of aE in the subduplicate ratio of the square of ba to the rectangle contained by ba and dp, that is, in the subduplicate ratio of fa to dp; and the velocities acquired by falling through fa and dp are in the same ratio. But the velocity in the cycloid at E is equal to that which would be acquired by falling from d through dp; consequently, the velocity of n in the semicircle dna is half of that which would be acquired by falling through fa. Hence, the time in which the body E describes any arch of the cycloid HE is the same in which the point n describes the arch dn by an uniform motion with half the velocity that would be acquired by falling through the perpendicular fa; and the time in which the body E descends along the arch Ha by falling from H to a, is equal to the time in which the point n with the same uniform motion describes the semicircle dna, which is equal to the time in which the point m describes the semicircle bma uniformly with half the velocity acquired by falling through ba, because dna is to bma as da is to ba, or in the subduplicate ratio of fa to ba; that

that is, in the ratio of the velocity acquired by falling through *fa* to that acquired by falling through *ba*. Therefore the time in which a body descends along the arch H*a* from any point H to the lowermost point *a* is equal to the time in the semicycloid AE*a*, and is the same where-ever the point H be in AE*a* from which it begins to descend, which is the celebrated property of the cycloid discovered by Mr. HUYGENS. The same property may be demonstrated by shewing, that the power by which the velocity of the body in the cycloid is accelerated at E is always as E*a* the arch from E to the lowermost point *a*; and an analogous property is shewn of the epicycloid (which is described by a given point in the plane of a circle that revolves upon a circular base) by Sir ISAAC NEWTON. Produce *an* till it meet the circle *bma* n *v*; and, if *ve* parallel to *n*E meet the cycloid in *e*, the arch HE shall be described when the motion begins from H in the same time that A*e* is described when the motion begins from A.

409. Rays of light being supposed to issue from a given point, and to be reflected by a given curve, so as to make the angle of reflexion equal to the angle of incidence, a curve that touches all
FIG. 182. the reflected rays is called the *Caustic by Reflexion.* Let S be the given point from which the rays issue, (which is therefore called the *focus* of the incident rays,) SL any incident ray, PL*p* the tangent at L, LC the ray of curvature at L, L*m* the reflected ray constituting the angle CL*m* equal to CLS; then, if the reflected rays always touch the curve *bme*, it is the caustic by reflexion. Let SP perpendicular to the tangent LP meet it always in P a point of the curve DP; let HME be the curve by the evolution of which DP is described according to art. 402. and let PM touch HME in M; join SM, and produce it to *m*, so that S*m* be equal to 2SM: then *m* shall be a point in the caustic of the curve BL, when S is the radiating point. For, because MP is perpendicular to the curve DP, the angle MPS is equal to the complement of SLP (by art. 211. & 212.) or to LSP, and SL is bisected by MP in K; therefore S*m* is to SM as SL is to SK, L*m* is parallel to KM the tangent of HME: but the figure *bme* is similar to HME, and similarly situated, (art. 122.) therefore L*m* is the tangent of *bme*. Because L*m* is parallel to PM, the angle *m*LC is equal to MPS, or LSP, or CLS;

CLS; therefore, when SL is the incident ray, L*m* is the reflected ray, and *m* is a point in the caustic.

410. Let L*ſ* be taken on the reflected ray equal to LS; and, CR being perpendicular from the center of curvature on this ray in R, biſect LR in *q*; and *qſ*, *q*R and *qm* ſhall be in continued proportion. For, by art. 390. *ſq*, *ſ*L and 2PM are in continued proportion. Let CI be perpendicular to SL in I, and IL be biſected in Q; when S is on the concave ſide of the curve, and LS is greater than LQ, PM is equal to the ſum of PK (or SK) and KM, and 2PM is equal to the ſum of *ſ*L and L*m*; therefore *ſq* is to *q*L or *q*R as *ſ*L is to L*m*, or as *q*R is to *qm*. When S is betwixt Q and L, 2PM is equal to the difference of L*m* and L*ſ*, and *ſq* is to *q*L as *ſ*L is to L*m*, or as *q*L (or *q*R) is to *qm*. In like manner it appears, that when S is on the convex ſide of the curve, *ſq*, *q*R (or *q*L) and *qm* are in continued proportion. When the incident ray is perpendicular to the curve, the reflected ray coincides with the incident ray, *ſ* with S, C with R, and *qm*, *q*C, *q*S are in continued proportion. In general, the rectangle *ſqm* is equal to the ſquare of *q*R, or *q*L; and, when the incident rays are parallel, the point *m* muſt coincide with *q*, and L*m* be equal to one half of LR. When LS is equal to one half of LI, and S is on the concave ſide of the curve, *ſ* coincides with *q*, and the reflected ray L*m* becomes an aſymptote of the cauſtic. Fig. 183.

411. Let BL be a circle as in art. 283. C the center, CB the radius that paſſes through S the radiating point; biſect BC in *q*, let *q*S, *q*C and *q*H be in continued proportion: and, when L comes to B, *m* ſhall come to H. When CS is leſs than one half of CB, the curve DP has no point of contrary flexure, and the cauſtic has no aſymptote. When CS is equal to one half of CB, the diameter through S is the aſymptote of the cauſtic. When CS is greater than one half of CB, but leſs than CB, the cauſtic meets CB produced beyond B, the part of the curve DP adjoining to B is convex towards S, P is a point of contrary flexure when SL is one fourth part of the chord LZ that paſſes through S, (as was ſhewn in art. 283.) and the reflected ray is then an aſymptote of the cauſtic. Fig. 105.

412. Let SA be a right line given in poſition, and L*m* ſhall Fig. 182.
be

be to SL as the fluxion of the angle ASL is to the fluxion of 2ASP -- ASL. Hence, in all the curves constructed in art. 392. & 393. *Lm* is to SL in an invariable ratio, because in those figures the angle ASL is to ASP in an invariable ratio; only, in the parabola, S being the focus, the fluxion of ASL is equal to the fluxion of 2ASP, and the reflected rays are parallel to each other and to the axis of the figure.

413. When the rays that issue from a given point are refracted at the curve, so that the sine of the angle contained by the refracted ray and the perpendicular to the curve is always to the sine of the angle contained by the incident ray and that perpendicular, in one constant ratio, the curve that touches all
FIG. 184. the refracted rays is called the *Caustic by Refraction.* Let S be the radiating point, SL any incident ray, LR the refracted ray, C the center of the curvature at L, CI perpendicular from C on the incident ray in I, CR perpendicular on the refracted ray in R; join SC and IR; let RZ perpendicular to RI meet CI in Z; join LZ meeting SC in Q, and let QM parallel to RZ meet the refracted ray LR in M: then shall M be in the caustic. For, supposing the refracted ray LR to touch the caustic in any point M, and LT parallel to CI to meet SC in T, and the point L in describing the curve BL to move towards *l*; then the angular velocity of SI about S shall be to the angular velocity of MR about M in the ratio compounded of the direct ratio of the fluxion of CI to the fluxion of CR (or the ratio of CI to CR, by art. 27. the ratio of CI the sine of the angle of incidence to CR the sine of the angle of refraction being supposed invariable) and of the inverse ratio of SI to RM. The angular velocity of SL about S is to the angular velocity of ML about M (by art. 208. & 202.) in the direct ratio of LI to LR (or of CZ to CR, because the triangles CZR, LIR are similar) and the inverse ratio of SL to LM. Therefore LM is to RM as the rectangle contained by CI and SL is to the rectangle contained by SI and CZ, or (because SI is to SL as CI is to LT) as LT is to CZ, or LQ to ZQ; consequently, QM and ZR are parallel, and the point M was rightly determined. When the incident rays are parallel, let CB parallel to those rays through the center of curvature meet

LZ

LZ in Q, and QM parallel to RZ shall meet the refracted ray LR in the caustic at M. When the incident ray becomes perpendicular to the refracting curve, that is, when L comes to B, the angular velocity of SL is to the angular velocity of ML as MB is to SB; therefore the caustic meets SC in M so that CM is to BM in the ratio compounded of the given ratio of CR (the sine of refraction) to CI (the sine of incidence) and of that of SC to SB: and when the incident rays are parallel, CM is to BM as the sine of refraction is to the sine of incidence. Fig. 185.

414. Describe the circle LICV upon the diameter LC, let CV parallel to the incident ray SI meet this circle in V; and RZ shall always pass through V, the center of curvature C and incident ray SL being given. Therefore the point M is determined by applying CR in this circle so that it may be to CI in the given ratio of the sine of refraction to the sine of incidence, joining VR that meets CI perpendicular from C to the incident ray in Z, joining LZ that meets SC in Q, and drawing QM parallel to VR. When LZ is parallel to SC, the refracted ray is the asymptote of the caustic. Fig. 184.

415. Let BL be a circle, and, the incident rays being supposed parallel, let the caustic touch the refracted ray LM in M a point of the circle; then, because LR is equal to RM, LQ and CI are bisected in Z and I: and, the triangles CRZ, LRI being similar, it follows, that CZ, or one half of CI, is to CR as LI is to LR. Therefore, if the invariable ratio of the sine of incidence to the sine of refraction be expressed by that of I to R, the square of I shall be to the square of 2R as the square of LI to the square of LR, and 4RR — II shall be to II as the difference of the squares of LR and LI, or of CI and CR, is to the square of LI; consequently, 3RR is to II — RR as the square of the radius CL is to the square of LI, or CV: And in this manner Sir Isaac Newton determines the position of the ray that, after a refraction at L, a reflexion at M, and a second refraction at G, defines the interior rainbow; for, the rays incident about L being refracted so as to touch the caustic near to M, a point in the circle BLM, and being thence reflected, they will emerge nearly parallel at G. If it be required that the rays refracted about L should be reflected at *l* in Fig. 186. Fig. 187.

the circle nearly parallel to each other; then, by art. 411. MR must be equal to M*l*, or one half of LR, ZQ must be one half of LZ, CZ one half of ZI, and therefore equal to one third part of CI. But LI is to LR as CZ is to CR, or as I is to 3R, and 9RR — II is to II — RR as the square of CI is to the square of LI; therefore 8RR is to II — RR as the square of the radius CL is to the square of LI, or CV; and hence the ray is determined which after a refraction at L, two reflections at *l* and G, and a second refraction at H, defines the exterior rainbow; for, if G*m* be one fourth part of GH, or of L*l*, the rays reflected at G shall touch a caustic formed by this second reflexion at *m*, and emerge parallel after their refraction at H.

416. Of the various problems that depend on the curvature of lines, none are more useful in philosophy than those which relate to the centripetal and centrifugal forces. In this doctrine it is supposed, that a body at rest never moves of itself, and that a body in motion never changes the velocity or direction of its motion of itself; but that every motion would continue uniform and its direction rectilineal unless some external force or resistance affected it. Hence, when a body at rest always tends to move, or when the velocity of any rectilineal motion is accelerated continually, or when the direction of a motion is continually varied and a curve line described, these are supposed to proceed equally from the influence of some power that acts incessantly; which may be measured either by the pressure of the quiescent body against the obstacle that hinders it to move in the first case, or by the acceleration of the motion in the second, or by the flexure of the curve described in the third case, due regard being had to the time in which these effects are produced, and the other circumstances, according to the principles of mechanics. Effects of the power of gravity of each kind fall under our constant observation near the surface of the earth; for the same power which renders bodies heavy while they are at rest, accelerates them when they descend perpendicularly, and bends their motion into a curve line when they are projected in any other direction than that of their gravity. But we have access to judge of the powers that act on the celestial bodies by

by effects of the last kind only; and it is of these chiefly we are to treat here.

417. As the velocity of a variable motion is measured by the space which would be described by it in a given time if it was continued uniformly for that time, (art. 4. 5. & 6.) and not by the space that is actually described by the variable motion in that time; so the power by which the rectilineal motion of a given body is continually accelerated or retarded is measured by the increment or decrement of the velocity that would be generated by that power in a given time, if its action, or influence, was continued uniformly for that time, and not by the increment or decrement of the velocity that is actually generated if the action of the power varies. The fluxion of the velocity is measured in the same manner as was explained in art 70. Therefore the power that accelerates or retards a rectilineal motion is always measured by the fluxion of the velocity of the motion, or (because the velocity is itself the first fluxion of the space) by the second fluxion of the space described by the motion, the time being supposed to flow uniformly. Thus, when a given body ascends, or descends, in the right line that is in the direction of its gravity, in spaces void of resistance, the power that accelerates or retards its motion at any term of the time (that is, the accelerating force of its gravity) is measured by the fluxion of the velocity; and if the velocity increase or decrease uniformly so that its fluxion be constant, the gravity must be supposed uniform. If the body descend, or ascend, in a medium that resists its motion, the power that accelerates or retards its velocity (that is, the difference of the gravity and resistance when the body descends, and their sum when it ascends) is still measured by the fluxion of the velocity, or the second fluxion of the space that is described by the motion, the time being supposed to flow uniformly.

418. Suppose the right line D*d* to move parallel to itself a- FIG. 188.
long the given line AO with an uniform motion, and the gravity to act always in the direction D*d*; then the body will descend in this line D*d* in the same manner as if the line was quiescent and the gravity acted in its direction. Therefore the gravity will be still measured by the fluxion of the velocity

with which the body defcends in the right line D*d*, or by the fecond fluxion of DE the fpace defcribed by it in this line, that is, by the fecond fluxion of the ordinate of the curve traced by the body on the immoveable plane AO*os*, the time (or the right line AD which flows in the fame manner as the time) being fuppofed to flow uniformly.

419. When a curve is defcribed by a gravity that acts in parallel lines, its force is as the fquare of the velocity of the body directly and that chord of the circle of curvature which paffes through the body in the direction of the gravity inverfely. For, the fame things being fuppofed as in the laft article, let the velocity of the point D or the fluxion of AD be reprefented by a given line DG, let GH parallel to the ordinate DE, meet the tangent ET in T; and the fluxion of the curve FE, or the velocity of the body in defcribing it, fhall be meafured by ET, by prop. 14. Let the circle of curvature at E meet E*d* in B, and the fecond fluxion of DE fhall be meafured by a right line that is a third proportional to $\frac{1}{2}$EB and ET, by prop. 33. Therefore the gravity, which is meafured by the fecond fluxion of DE, is as the fquare of ET directly and EB inverfely, that is, as the fquare of the velocity directly and the chord of the circle of curvature which paffes through E in the direction of the gravity inverfely.

420. This may be further illuftrated, if it feem neceffary, in the following manner. If the gravity act uniformly and in parallel lines upon a body that fets out from E in the direction ET, and bend its courfe into the curve ER, and the right lines TR, V*x* in the direction of the gravity meet ET in T and V, and meet the curve in R and *x*, V*x* fhall be to TR as the fquare of EV is to the fquare of ET, by what was fhewn in art. 95. confequently, the rectangle contained by V*x* and an invariable right line is equal to the fquare of EV, and the curve E*x*R is a parabola. And it may be fhewn converfely, from art. 254. that when a parabola E*x*R is defcribed by a gravity that acts always in a direction parallel to the axis of the figure, then, DG being given, TR is invariable; and, confequently, the gravity is uniform. Let HE*b* be any curve line that touches the parabola E*x*R and has the fame curvature with it at E; then, if the power by

by which EH is defcribed act always in the fame direction as the uniform gravity by which the parabola ER is defcribed, and the velocities at E in EH and ER be equal, the gravity in the curve *b*EH at E muft be equal to the uniform gravity in the parabola ER. For, if the gravity in EH at E be faid to be greater than the uniform gravity in the parabola ER, let it be greater in the ratio of TZ to TR; and let EZ be a parabola defcribed through E and Z that has the fame tangent ET and diameter E*d* with the parabola ER at E. Then the gravity in the curve EH at E fhall be equal to the uniform gravity in the parabola EZ; and, if the variable gravity in EH be fuppofed firft to increafe while the body moves from E to H, TZ fhall be always lefs than TH, becaufe a fpace defcribed by a power that acts uniformly muft be lefs than the fpace which is defcribed in the fame time when that power continually increafes from the beginning of the time. But TR is always lefs than TZ; therefore the parabola EZ paffes between the curve EH and parabola ER. But EH and ER were fuppofed to have the fame curvature at E; confequently, no parabola can pafs between them, by art. 371. And thefe being contradictory, it follows, that, when the gravity in EH increafes from E to H, the gravity at E in EH cannot be greater than the uniform gravity in the parabola ER. If the gravity in EH be fuppofed to decreafe from E to H, let a parabola E*u* be defcribed through any point *u* between R and Z, fo as to have the fame tangent and diameter at E with ER and EZ; then, fince the gravity in EH at E is equal to the uniform gravity in the parabola EZ, (by the fuppofition) it muft be greater than the uniform gravity in the parabola E*u* which is greater than the uniform gravity in ER. Therefore the part of the parabola E*u* that is adjoining to E paffes between the curve EH and parabola ER; and (by art. 371.) ER has not the fame curvature with EH, againft the fuppofition. In like manner it may be fhewn, that the gravity in EH at E is not lefs than the uniform gravity in the parabola ER. Therefore they are equal to each other. Let the common circle of curvature meet E*d* in B, and TR, ET and EB fhall be in continued proportion, by art. 371. and TR is as the fquare of ET directly and EB inverfely. Therefore the gravity in EH

at

at E is as the ſquare of the velocity at E (which is meaſured by ET, DG being given) directly and as the chord of the circle of curvature through E in the direction of the gravity inverſely; for the uniform gravity in the parabola ER is meaſured by 2TR, by art. 74. & 75.

421. When the velocity and direction of the motion and the force and direction of the gravity at E are given, the curvature at E is given. For, let ET be the ſpace that would be deſcribed in a given time by the motion at E continued uniformly, let TR be equal to the ſpace deſcribed in the ſame time by a body falling from E in the right line E*d* when the gravity at E is continued uniformly; then EB the chord of the circle of curvature ſhall be a third proportional to TR and ET; and, ſince the angle BET is given, the center and ray of curvature are given. The curvature therefore of EH at E depends only on the force and direction of the gravity at E, the velocity and direction of the projection at E being given, and not on the ſubſequent variations of the gravity. When the force and direction of the gravity with the velocity at E are given, EB is determined, and the ray of curvature is reciprocally as the ſine of the angle BET.

422. Let EH be now deſcribed by a centripetal force directed towards a given point S in the right line EB; and let ER be the parabola that would be deſcribed by the ſame centripetal force at E continued uniformly in a direction always parallel to EB: then the contact of the curve EH (that is deſcribed with a variable centripetal force directed towards S) with the parabola ER (that is deſcribed with the centripetal force in EH at E continued uniformly from that term) ſhall be cloſer than the contact of EH with a parabola that is deſcribed by a greater or leſs force continued uniformly in the ſame direction. Therefore the curve EH and parabola ER have the ſame curvature at E. From which it follows, that the centripetal force in the curve EH at E directed towards any point S is as the ſquare of the velocity of the motion at E directly and EB the chord of the circle of curvature that paſſes through S the center of the forces inverſely.

423. Let SE be produced till it meet the right line AD given in poſition; and, if AD be ſuppoſed to flow uniformly, the centripetal

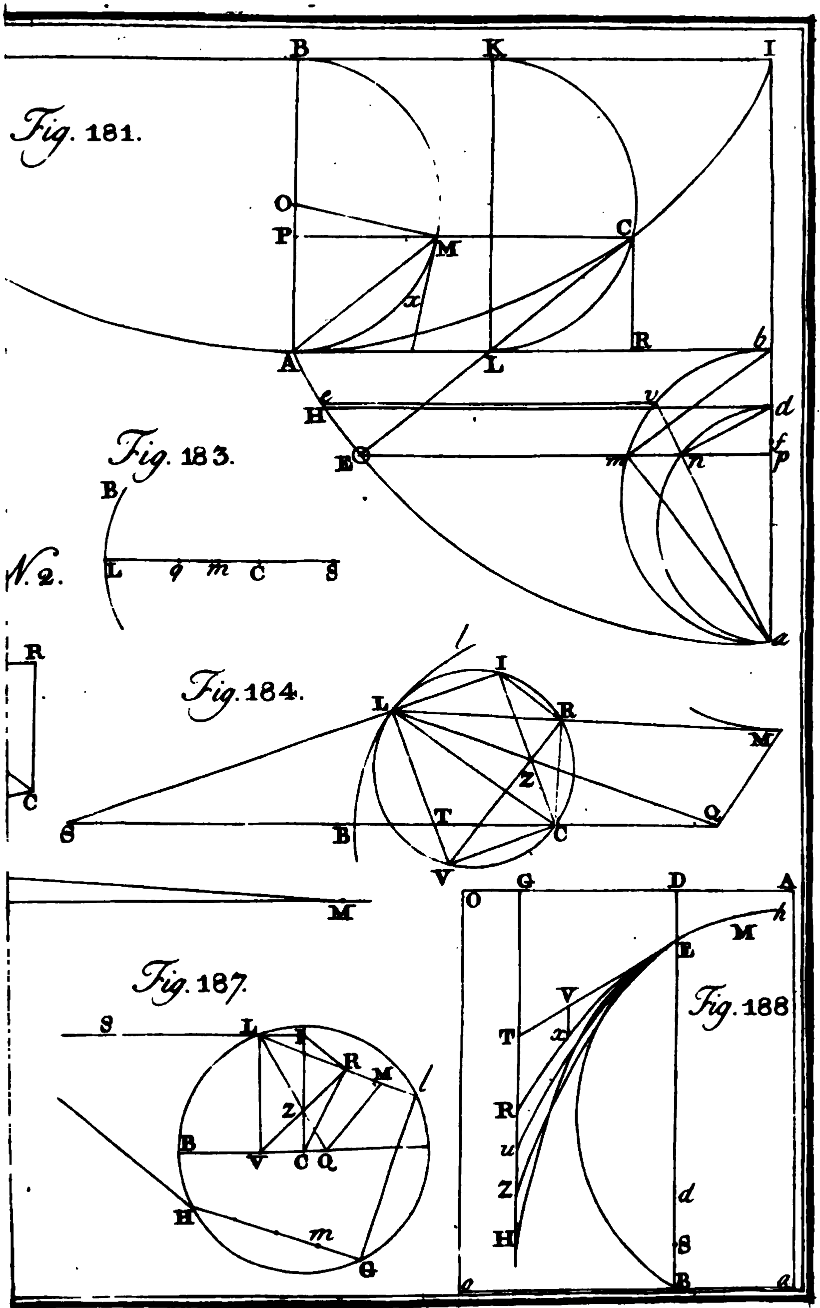


Fig. 181.

Fig. 183.

N. 2.

Fig. 184.

Fig. 187.

Fig. 188

tripetal force ſhall be meaſured by the ſecond fluxion of the ordinate DE, if the center S to which this force is directed be any where in the right line EB produced from E on the concave ſide of the curve; becauſe the ſecond fluxion of DE conſidered as the ordinate from the curve EH is equal to the ſecond fluxion of DE conſidered as the ordinate from the parabola ER that has the ſame curvature with EH at E, by art. 383. In the ſame manner it appears, that, if the curve EH be deſcribed by a centrifugal force directed to any point in the right line EB produced from E on the convex ſide of the curve, this force ſhall be likewiſe meaſured by the ſecond fluxion of the ſame ordinate.

424. Since the ſquare of ET is equal to the rectangle contained by TR and EB, it follows, that the velocities at any two points of curve lines are to each other in the ratio compounded of the ſubduplicate ratio of the centripetal forces at theſe points, and the ſubduplicate ratio of the chords of the circles of curvature which paſs through theſe points and the centers of the forces. When the gravity is given, the velocities are in the ſubduplicate ratio of theſe chords. Therefore the velocity at any point E is to the velocity of a body that deſcribes a circle about S in a void at the ſame diſtance SE by the ſame centripetal force in the ſubduplicate ratio of EB to 2SE, becauſe the chord of the circle that paſſes through the center is its diameter and is equal to 2SE; conſequently, if SP be always perpendicular from S on the tangent ET, the velocity in the curve EH at E ſhall be to the velocity in ſuch a circle in the ſubduplicate ratio of the angular velocity of SE about S to the angular velocity of SP about S; becauſe, by art. 389. theſe angular velocities are to each other in the ratio of EB to 2SE. Thus, for example, the velocity in a parabola, when it is deſcribed by a centripetal force directed towards the focus, is to the velocity in a circle that is deſcribed by the ſame centripetal force at the ſame diſtance in the ſubduplicate ratio of 2 to 1; becauſe, if A be the vertex of the parabola, the angle ASE is always double of ASP. In the logarithmic ſpiral theſe velocities are always equal, becauſe the angular velocities of SE and SP are equal in it, by art. 349. In all the figures conſtructed in art. 392. & 393. theſe velocities are to each other in an invariable ratio when the cen- FIG. 189.

tripetal

Fig. 171. & 172. tripetal force is directed towards S, *viz.* in the subduplicate ratio of n to $n-1$ in those of art. 392. and in the subduplicate ratio of n to $n+1$ in those of art. 393. because the angular velocities of SL and SP about S are to each other in the same invariable ratio as the angles ASL, ASP in those figures.

Fig. 158. 425. When a conic section is described by a centripetal force directed towards any point S in the plane of the section, let O*a* the semidiameter parallel to the tangent at E meet SE in R; and the velocity in the section at E shall be to the velocity in a circle described in a void at the same distance SE by the same centripetal force as O*a* is to a mean proportional betwixt SE and RE. For, by the fourth property of the circle of curvature demonstrated in art. 375. the rectangle RE*b* (E*b* being one half of EB) is equal to the square of O*a*, and EB is to 2SE as the square of O*a* is to the rectangle RES. When the center of the forces coincides with O the center of the figure, the velocity in the conic section is to the velocity in the circle as O*a* to OE; because, in this case, SE and RE coincide with OE. When S is in the focus of the section, these velocities are to each other in the ratio of O*a* to a mean proportional between the distance SE and half the transverse axis of the figure, (which in this case is equal to RE,) or in the subduplicate ratio of the distance of the revolving body from the other focus to half the transverse axis; and these velocities are equal at the extremity of the shorter axis in the ellipse.

426. The velocity and direction of the motion and the centripetal force at any given point E, with the center O of the conic section, towards which the force is supposed to be directed, being given, the section is determined that can be described by such a centripetal force. For, if T*r* be equal to the space that a body falling from E in the right line OE would describe by its gravity at E continued uniformly in the same time that ET would be described by the motion in the curve at E continued uniformly, a third proportional to T*r* and ET shall be equal to the parameter of the diameter that passes through E; therefore the semidiameter O*a* parallel to ET is given in position and magnitude, and the conjugate semidiameter OE being likewise given, the conic section is determined.

427. When

427. When the conic section is described by a centripetal force that is directed towards the focus S, let T*x* be the space that would be described by a body falling from E in the right line ES with the gravity at E continued uniformly, in the same time that ET would be described by the motion of the body at E continued uniformly in the tangent; let EB be to ET as ET is to T*x*, and EQ be equal to one fourth part of EB: then a third proportional to SQ and SE shall be equal to the transverse axis of the figure. Let SP be perpendicular from S on the tangent ET in P, bisect SE in *r*, join P*r*, let the point O be taken upon the right line P*r* (the same or the contrary way from P with *r* according as ES is greater or less than EQ) so that PO may be equal to one half of the transverse axis, and O shall be the center of the figure, by what was shewn in art. 375. When EQ is equal to ES, the figure is a parabola, the vertex of which is determined in that article; when EQ is less than ES, the figure is an ellipse; and when EQ is greater than ES, it is an hyperbola. It appears likewise, that, when the centripetal force and velocity at E with the distance SE are given, the transverse axis is given, whatever the direction of the motion at E (or the angle SET) may be. Fig. 159.

428. Let EH be any curve described by a centripetal force directed towards S; let the force at E be represented by the right line EK, and KV be perpendicular to the tangent ET in V; then EV shall represent the force by which the velocity of a body is accelerated in a void if it is projected from E towards T, or retarded if it is projected towards *t*; and KV shall represent the force by which its course is inflected from the tangent ET, or E*t*. Because KV is to EK as E*b* is to the ray of curvature EC, and EK is as the square of the velocity at E directly and E*b* inversely, it follows, that KV is as the square of the velocity at E directly and the ray of curvature EC inversely, and that the curvature of EH at E depends on the velocity at E and the force KV only. Hence, if the force EV be increased or diminished by any new force that acts in the direction of the tangent, and is in any assignable ratio to the force EK, but the force KV remain, the curvature at E of the arch EH described by the body shall remain the same as if that new force had not acted. Fig. 189.

acted. If we suppose the same curve to be described by a centripetal force directed towards any other point *f*, and the velocity at E to be the same as formerly, let K*k* and S*l* parallel to the tangent E*t* meet *f*E in *k* and *l*, and C*d* be perpendicular to *f*E in *d*, then the centripetal force towards *f* shall be to the centripetal force that was directed towards S in the former case as E*b* is to E*d*, (art. 422.) that is, as E*k* is to EK, (because the angle *d*E*t* is equal to E*bd*,) or as E*l* is to ES.

429. It follows from what is shewn in the last article, that, when a body describes any curve EH by a centripetal force directed towards S, in a *medium* the resistance of which is in an assignable ratio to the centripetal force, this force is still at E as the square of the velocity directly and the chord of the circle of curvature that passes through S inversely. For the resistance affects the force EV only which is in the direction of the tangent or of the motion of the body, and the velocity in the curve is accelerated or retarded in this case by the difference or sum of the force EV and the resistance; but the force KV that is perpendicular to the tangent (upon which the curvature of EH at E depends) remains the same as when the motion is in a void; and, since the curvature of EH at E is the same in the medium as in the void when the velocity and centripetal force at E are the same, it follows, that the centripetal force at E in the medium as well as in a void is as the square of the velocity at E directly and E*b* inversely. The velocity in the curve at E in the medium is to the velocity in a circle described in a void by the same centripetal force at the same distance, in the same ratio as if the curve was described in a void by a centripetal force directed to the same center S, that is, in the subduplicate ratio of the angular velocity of SE to the angular velocity of SP about S, by art. 424. Therefore what has been shewn in the preceeding articles on this subject is to be extended equally to both cases; and for this reason we chose first to shew how the velocity in the curve is compared with the velocity in a circle described by the same centripetal force at the same distance, before we should enquire into the variations of the velocity and centripetal force in a given figure, which are different in a medium from what they are in a space void of resistance. Thus,

if

if a logarithmic ſpiral, for example, be deſcribed either in a medium or in a void by a force directed to the center S, the velocity is always equal to the velocity in a circle deſcribed in a void at the ſame diſtance by the ſame centripetal force: And in any of the figures conſtructed in art. 392. & 393, theſe velocities are in an invariable ratio when S is the center of the forces.

430. It follows, that the centripetal forces at E directed towards a given point S by which the ſame curve EH can be deſcribed in a medium and in a void, are to each other in the duplicate ratio of the velocity at E in the medium to the velocity at the ſame point in a void; becauſe the chord of curvature is the ſame in both caſes.

431. Another theorem by which the centripetal forces may be diſcovered follows from what was ſhewn in art. 385. Let EV a circle deſcribed from the center S meet PM in V; and the ſecond fluxion of PM ſhall be equal to the difference of the ſecond fluxions of SM and PV when the points M and V ſet out together from E. Therefore, if we ſuppoſe that while the body M deſcribes the curve HEM by a force directed towards S, another body V revolves in the circle EV, the uniform angular velocity of SV about S being equal to the angular velocity of SM about S when M ſets out from E, and that a third body L deſcends or aſcends in the right line SE ſo that SL is always equal to SM; then the centripetal force of M in the curve HEM at E ſhall be equal to the difference of the centripetal force by which V is retained in the circle EV and of the force by which the motion of L is accelerated or retarded at E in the right line SE. For the centripetal force in the circle EV is meaſured by the ſecond fluxion of PV when V ſets out from E, AD being ſuppoſed to flow uniformly, by art. 423. and the force by which the motion of L is accelerated or retarded in the right line SE is meaſured by the ſecond fluxion of SL, or SM, when L comes to E, by art. 418. Therefore the difference of thoſe forces is meaſured by the difference of theſe ſecond fluxions of PV and SM, or the ſecond fluxion of PM, and, conſequently, is equal to the centripetal force in HEM at E, by art. 423. The centripetal force in the circle EV is e- Fig. 166.

 qual

qual to the centrifugal force that arifes from the motion of rotation in the fame circle, or in the curve EM at E, fince (by the fuppofition) the angular velocities of SM and SV are equal when M and V fet out from E. And hence it appears, that the force which accelerates or retards the velocity of L at E in the right line ES (which is fometimes called the paracentric velocity of M at E) is equal to the difference betwixt the centripetal force in EM at E and the centrifugal force in the circle EV, and not to the difference betwixt the centripetal force in EM
FIG. 190. at E and twice the centrifugal force in the circle EV. It follows from this, that, when two curve lines EM, *em* are defcribed by centripetal forces directed towards S, and SM is always equal to S*m*, the difference of thefe forces at E and *e* muft be equal to the difference of the centripetal forces in the circles EV, *ev*, the angular velocities of SV and S*v* being refpectively equal to the angular velocities of SM and S*m* when M and *m* fet out from E and *e*. And, if the velocity in the circle EV be to the velocity in the circle *ev* as G is to F, the difference of the forces in EM and *em* at E and *e* fhall be to the force in the circle EV as the difference of the fquares of G and F is to the fquare of G.

432. When a circle is defcribed by a centripetal force that is directed towards its center, the motion is uniform; for, the direction of the centripetal force being always perpendicular to the tangent, it neither accelerates nor retards the velocity of the body, and has no other effect but to bend its courfe continually from the tangent into the circle. The centripetal forces are as the fquares of the velocities directly and the diameters of the circles inverfely, by art. 422. or (becaufe the velocities are as the diameters directly and the times in which the revolutions are completed inverfely) as the diameters directly and the fquares of the periodic times inverfely. If rays be fuppofed to be drawn always from the centers to the revolving bodies, the centripetal forces fhall be likewife in the ratio compounded of the ratio of the angular velocities of thofe rays (or of the angular velocities of the tangents at the bodies) and the ratio of the velocities of the revolving bodies. Thus, when the circles are defcribed in equal periodic times, or the angular velocities

of

of thofe rays are equal, the velocities and the centripetal forces are as the diftances of the revolving bodies from the centers of the circles. When the fquares of the periodic times are as the cubes of the rays of the circles, and, confequently, the velocities in the fubduplicate ratio of thefe rays inverfely, the centripetal forces are reciprocally as the fquares of the rays. When the periodic times are as the fquares of the rays, or the velocities reciprocally as the rays, the centripetal forces are reciprocally as the cubes of the rays. In general, when the velocities are reciprocally as any power of the rays whofe exponent is m, the centripetal forces are reciprocally as the power of thofe rays whofe exponent is $2m + 1$: And, converfely, when the centripetal forces are reciprocally as the power of the rays whofe exponent is n, the velocities are reciprocally as the power of the rays whofe exponent is one half of $m - 1$.

433. The velocity in a circle is to the velocity that would be acquired by falling in a right line from the circumference to the center, by the fame centripetal force with which the circle is defcribed continued uniformly, as the radius is to the fide of the infcribed fquare, or as unit is to the fquare root of two. Let S be the center, SA the radius of the circle, and fuppofe the Fig. 191. centripetal force at A to continue to act upon the body uniformly and in parallel lines; then fhall it defcribe a parabola AR that has the fame curvature with the circle at A, or that has its focus in the point F, if SA be bifected in F, (by art. 422. & 371.) Let GA be equal to AF, and it is known that the velocity in this parabola at A is equal to the velocity that would be acquired by falling with the fame uniform centripetal force from G to A; and therefore is to the velocity that would be acquired by falling from A to S with the fame centripetal force in the fubduplicate ratio of GA to AS, (art. 95.) or of 1 to 2; and, fince the velocity in the circle is fuppofed equal to the velocity in the parabola in A, it muft be to the velocity acquired by falling from A to S in the fame ratio of 1 to the fquare root of 2. Hence the periodic time in a circle is to the time in which a body would fall directly from the circumference to the center by the fame gravity, as the circumference of the circle is to the fide of the infcribed fquare. In general, it appears in the fame manner, that

that the velocity in any curve at E is equal to the velocity that would be acquired in a void by falling directly through one fourth part of the chord of the circle of curvature EB, by the gravity at E continued uniformly; or that E*b* is always double of the ſpace in falling through which the velocity at E would be generated in a void by the gravity at E continued uniformly.

FIG. 21. 434. It was ſhewn in art. 95. that, when a motion is uniformly accelerated, that is, when the force that accelerates it is conſtant, if *ad* be the ſpace deſcribed by the motion and the triangular area ADE be always equal to the rectangle *ae* contained by *ad* and the invariable right line *af*, the time of the motion ſhall be repreſented by the baſe AD, and the velocity by DE. The force that accelerates the motion is as IH the fluxion of the velocity directly and EI the fluxion of the time inverſely, (by art. 114. or 417.) and therefore may be repreſented by a right line *ab* that is to the invariable right line *af* as IH is to EI, or DE to AD, or as the ſquare of DE is to the rectangle ADE which is equal to twice the rectangle *fad*; conſequently, the ſquare of DE which meaſures the velocity at D is equal to twice the rectangle *bad* contained by *ba* which meaſures the centripetal force and *ad* the ſpace deſcribed from the beginning of the motion; and the fluxion of the ſquare of the velocity is meaſured by the double rectangle contained by *ba* and the right line which meaſures the fluxion of the ſpace.

FIG. 191. Hence, if a body be ſuppoſed to fall in the right line *a*S by a centripetal force in the direction *a*S that at any diſtance SQ is meaſured by the ordinate QN, the fluxion of the ſquare of the velocity at Q ſhall be meaſured by twice the rectangle contained by QN and the right line which meaſures the fluxion of *a*Q: but this rectangle meaſures the fluxion of the area *a*QN*d*, by prop. 3. therefore the ſquare of the velocity acquired by falling from *a* to Q is meaſured by twice the area *a*QN*d*, by art. 24. This, if it was neceſſary, might be demonſtrated in the ſame manner as the third propoſition, from this principle, which is analogous to the firſt axiom, That when a motion is accelerated by a centripetal force that increaſes continually, the velocity generated by it while a given ſpace is deſcribed, is greater than the velocity which would have been generated if the accelerating

ting force had not increaſed, but continued uniform. It is likewiſe obvious, that if *a*Q was a column of an homogeneous fluid, and the gravitation of any particle at the diſtance SQ was repreſented by the ordinate QN, the preſſure of the fluid at Q would be as the area *a*QN*d*, or as the ſquare of the velocity that would be acquired by falling from *a* to Q.

435. Let the velocity in the curve AE at A be equal to that which would be acquired by falling from *a* to A in the right line *a*S, ſo that its ſquare may be repreſented by twice the area *a*AD*d*; let SE be equal to SQ and the centripetal force at E be ſuppoſed equal to the centripetal force at Q which is repreſented by the ordinate QN; let EK be equal to QN, and KV be perpendicular to the tangent EP in V: then the force by which the motion in the curve at E is accelerated ſhall be repreſented by EV, and the fluxion of the ſquare of the velocity at E by twice the rectangle contained by EV and the right line which meaſures the fluxion of the curve AE, by the laſt article: But this rectangle is equal to the rectangle contained by EK, or QN, and the right line which meaſures the fluxion of *a*Q, becauſe (by prop. 17.) the fluxion of the curve AE is to the fluxion of *a*Q as EK is to EV; therefore the fluxion of the ſquare of the velocity at E is meaſured by twice the fluxion of the area *a*QN*d*; and the ſquare of the velocity at E is meaſured by 2*a*QN*d*. The ſquare of the ſame velocity is meaſured by twice the rectangle contained by QN and $\frac{1}{4}$EB, (by art. 433.) or by the rectangle contained by QN and E*b*; conſequently, the rectangle contained by QN and E*b* is double of the area *a*QN*d*: and hence, when the centripetal force at E, or Q, is known, and the law according to which this force varies in different diſtances, E*b* is found by the quadrature of the area *a*QN*d*. Let the ſquare of the right line QR be equal to twice the area *a*QN*d*, that QR may repreſent the velocity at E; then QN, QR and E*b* ſhall be in continued proportion; and, the fluxion of the ſquare of QR being equal to twice the fluxion of the area *a*QN*d*, the fluxion of QR ſhall be to the fluxion of *a*Q as QN is to QR, or as QR is to E*b*; that is, the fluxion of QR which repreſents the velocity at E is to the fluxion of *a*Q, or S*a* — SE, as QR is to E*b* half of the chord EB.

436. Let

436. Let the ordinate AD represent the centripetal force at the diſtance SA, and the velocity in a circle deſcribed about the center S at the diſtance SA with the centripetal force AD ſhall be to the velocity acquired by falling from a to Q in the ſubduplicate ratio of the triangle SAD to the area aQNd, by art. 433. & 434. And when a body is projected upwards from A in the right line Aa with a velocity equal to that in the circle, it will juſt riſe to the point a, and return again from thence, if the area ADad be equal to the triangle SAD. When the centripetal force is reciprocally as any power of the diſtance whoſe exponent is any number m greater than unit, there is a limit which always exceeds the area ADda how great ſoever the height Aa may be, by art. 293. & 325. and this limit is to the rectangle SAD (or twice the triangle SAD) as unit is to $m - 1$. Hence, if a body be projected upwards from A in the right line Aa with a velocity that is to the velocity in a circle deſcribed at the diſtance SA with the centripetal force AD in the ſubduplicate ratio of 2 to $m - 1$, or in any greater ratio, the body will riſe for ever in the right line Aa, and never return again to A. If the velocity of a body that moves in the right line Sa, or in any curve AE, be at any point A to the velocity in a circle at the diſtance SA in the ſubduplicate ratio of 2 to $m - 1$, its velocity will be always to the velocity in a circle at the ſame diſtance from S the center of the forces in the ſame ratio, becauſe the ratio of the triangle SAD to the limit of the area ADda continued upwards from A is always the ſame where-ever the point A be taken in the right line Sa. This velocity that is neceſſary to carry off a body for ever in the right line Aa is greater or leſs than the velocity in a circle at the ſame diſtance according as 2 is greater or leſs than $m - 1$, that is, according as m is leſs or greater than 3; and theſe velocities are equal when m is equal to 3. When a body is projected with this velocity in any other direction than that of its gravity, Eb is to SE as 2 is to $m - 1$, by art. 435. when m is leſs than 3, the body riſes for ever in one of thoſe parabolic figures that were conſtructed in art. 392. (Fig. 171.) when m is greater than 3, it approaches to the center till it fall into it in one of the figures that were conſtructed in art. 393. (Fig. 172.) Firſt, let

m be less than 3, let L be any point in the trajectory, LP the tangent, SP perpendicular to LP in P, and let n be to unit as 2 is to the excess of 3 above m; let the angle LSA be made on the same side of LS with LSP, and in the same ratio to LSP as n is to unit; let SA be to SL as the power of SP whose exponent is n is to the same power of SL: then shall A be the *Apsis* of the figure, or the point where the tangent becomes perpendicular to the ray drawn from the center of the forces to the point of contact. Let AM be a right line perpendicular to SA, and the trajectory AL may be constructed by this right line AM, as in art. 392. by drawing SM from S to any point in this right line, making always the angle ASL to ASM as n is to unit, and SL to SA as the power of SM whose exponent is n is to the same power of SA. The demonstration is no more than the converse of what was shewn in that article. For, supposing L to be any point in the trajectory, A the apsis where SL and SP coincide; and, since the velocity at L is always to the velocity in a circle described at the same distance by the same centripetal force in the subduplicate ratio of 2 to $m-1$, by what has been shewn, it follows, (by art. 424.) that the fluxion of the angle ASL is to the fluxion of the angle ASP, and the angle ASL to ASP (art. 33.) as 2 is to $m-1$; so that ASL must be to PSL as 2 is to $3-m$, or n to unit; consequently, if we suppose AM to be perpendicular to SA, and the angle ASM to be to ASL as $3-m$ is to 2, the angles ASM, PSL shall be equal, and SP to SA as SL is to SM. Let the arches fL, kP described from the center S through L and P meet SA in f and k, and the fluxion of SL shall be to the fluxion of SP as the fluxion of the arch fL is to the fluxion of kP, (by art. 211. and 202.) that is in the ratio compounded of that of SL to SP and that of 2 to $m-1$: from which it follows, (by the converse of art. 167.) that SP is to SA (or SL to SM) as the power of SL whose exponent is one half of $m-1$ is to the same power of SA, or that SL is to SA as the power of SM whose exponent is n is to the same power of SA; and that the trajectory is constructed in the manner we have described from art. 392. Because the velocity of projection at A is greater than the velocity by which a circle is described at the same di- Fig. 171.

 stance

ſtance with the ſame centripetal force, the body begins to recede from the center at the apſis A, and it goes off in a parabolic curve by the conſtruction: When m is equal to 2, for example, the angle ASL is double of ASM, SA, SM and SL are in continued proportion, and AL is a common parabola that has its focus in S the center of the forces. Becauſe ASM never amounts to a right angle, ASL is to a right one in a ratio that is always leſs than that of 2 to $3 - m$, or that of n to unit: but if we ſhould ſuppoſe (as is uſual) the body to go off to infinity in the trajectory AL, the angle ASL deſcribed by the ray SL ſhall be to a right angle in that ratio; or, if the exceſs of 3 above m be expreſſed by the fraction $\frac{1}{s}$, the body will go off in a number of revolutions denoted by $\frac{1}{2}s$. Thus, if the centripetal force be reciprocally as the power of the diſtance whoſe exponent is $2\frac{99}{100}$, and the velocity of the projection at A be to the velocity in a circle deſcribed at the ſame diſtance with the ſame centripetal force in the ſubduplicate ratio of 200 to 199, the body will go off in 50 revolutions. If m be ſuppoſed ſucceſſively equal to $1\frac{1}{2}$, 2 and $2\frac{1}{2}$, and the velocity of the projection be to the velocity in a circle at the ſame diſtance in the ſubduplicate ratio of the number 4 to 1, 2 or 3, the body will go off in $\frac{1}{3}$, $\frac{1}{2}$, or one revolution, reſpectively. The body goes off in an hyperbolic curve when the ratio of thoſe velocities is greater than the ſubduplicate ratio of 2 to $m - 1$; and when the ratio of thoſe velocities is leſs, the body revolves continually about S within certain limits.

FIG. 172. 437. When m is greater than 3, and the velocity of projection at L is to the velocity in a circle deſcribed by the ſame centripetal force at the ſame diſtance in the ſubduplicate ratio of 2 to $m - 1$, let n be to unit as 2 is to the exceſs of m above 3, make the angle LSA in the ſame ratio to LSP as n is to unit, but on the oppoſite ſide of SL; let SA be to SL as the power of SL whoſe exponent is n is to the ſame power of SP; and A ſhall be the apſis of the trajectory. Upon the diameter SA deſcribe a ſemicircle SMA, and by it the curve AL is to be conſtructed (as in art. 393.) by drawing any right line as SM to the ſemicircle, making always the angle ASL to ASM as n is

is to unit, and SL to SA as the power of SM whose exponent is n is to the same power of SA: The demonstration is similar to that in the last article. When the centripetal force is reciprocally as the fourth power of the distance, m is equal to 4; and if the velocity of the projection be to the velocity in a circle at the same distance as the square root of 2 is to the square root of 3, then n is equal to 2, the angle ASL is double of ASM, the right lines SA, SM and SL are in continued proportion, and the trajectory AL is an epicycloid that is described by a point in the circumference of a circle while it revolves upon an equal circle, as we have shewn elsewhere. When m is equal to 5, 2 is to $m-1$ as 1 to 2; and if the velocity of the projection be to the velocity in the circle as 1 to the square root of 2, n is equal to unit, and the semicircle itself is the trajectory. When m is equal to 7, 2 is to $m-1$ as 1 to 3; and if the velocity of the projection be to the velocity in the circle as 1 to the square root of 3, n shall be equal to $\frac{1}{2}$, or ASL shall be one half of ASM, SL shall be a mean proportional betwixt SA and SM, and the trajectory is the same figure that is called the *lemniscata* by the celebrated Mr. BERNOULLI. It is manifest that the body falls always into the center S in a number of revolutions from the apsis A denoted by $\frac{1}{4}n$, or by $\frac{1}{2}s$ if the fraction $\frac{1}{s}$ be equal to the excess of m above 3. Thus, if m be equal to $3\frac{1}{100}$, and the velocity of projection at A be to the velocity in the circle at A in the subduplicate ratio of 200 to 201, the body will fall into the center after 50 revolutions; and the number of such revolutions is inversely as the excess of m above 3. The same constructions of the trajectories in this and the preceeding article were given long ago, *Descript. curv. par.* 2. *prop.* 22.

438. When m is equal to 3, and the velocity of the projection is equal to the velocity by which a circle is described at the same distance with the same centripetal force, the angular velocity of SE is equal to the angular velocity of SP; therefore the angles ESP, SPE are invariable, and the trajectory is the logarithmic spiral in which the body will approach to the center S, or recede from it, according as the direction of the FIG. 191.

 pro-

projection forms an acute or an obtuse angle with the ray drawn to the center S.

439. When m is equal to unit, or less, there is no assignable space which the area ADda may not exceed by producing Aa upwards, (by art. 325.) and a body projected upwards from A will return again how great soever the velocity of the proje-
FIG. 192. ction may be. If we suppose a centrifugal force to be always directed from the point S that is inversely as the power of the distance whose exponent is any number m less than unit, and a body to be projected at any point L with the velocity which it would acquire by this centrifugal force in moving from S to L, or that is to the velocity in a circle described at the same distance by a centripetal force equal to this centrifugal force in the subduplicate ratio of 2 to $1-m$, the trajectory will be constructed by the right line AM in the same manner as in art. 436. only, in this case, n is a fraction that is to unit as 2 is to $3-m$. Thus, if the centrifugal force be invariable, and the ratio of these velocities be that of the square root of 2 to unit, the trajectory will be constructed by taking the angle ASL always equal to two thirds of the angle ASM and SL equal to the first of two mean proportionals betwixt SM and SA; for in this case n is to unit as 2 is to 3. If the centrifugal force be as the distance, n is equal to $\frac{1}{2}$, ASL is one half of ASM, SL is a mean proportional betwixt SM and SA, and the trajectory is an equilateral hyperbola that has its center in S. In all those cases the trajectory has an asymptote that passes through S and constitutes with SA an angle which is to a right one as 2 is to the excess of 3 above m.

440. It may be of use, to avoid mistakes that may arise in enquiries of this kind, and in other cases where second fluxions are introduced, to reflect here on what was shewn concerning these fluxions above, in art. 74. 97. and in several other places.
FIG. 191. The centripetal force in the curve AEM at E is most commonly measured by TM the subtense of the angle of contact at E when the tangent ET is supposed to be diminished infinitely, the time in which the arch EM is described being given; that is, by Tr when ET which represents the velocity at E is of a finite magnitude (as we always suppose it) and

Er

Er is a parabola of the ſame curvature at E with AEM: becauſe Tr is the ſpace which the body would deſcribe by the gravity at E continued uniformly, in the ſame time that the body would deſcribe ET by its motion at E continued uniformly in the tangent. If therefore we ſuppoſe QN (which repreſents the centripetal force at the diſtance SQ, or SE) to be equal to Tr, the ſquare of the velocity at E will be repreſented by the rectangle contained by EB (or $2Eb$) and QN, becauſe the ſquare of ET is equal to the rectangle contained by EB and Tr. And, ſince the ſquare of the velocity that is acquired by falling through Eb with the ſame gravity QN continued uniformly, is repreſented by twice the rectangle contained by Eb and QN, (art. 434.) it ſhould follow, that the velocity in the curve at E is equal to the velocity that would be acquired in falling through Eb by the gravity at E continued uniformly; whereas it has been ſhewn, (art. 433.) that theſe velocities are not equal, but are to each other in the ſubduplicate ratio of 1 to 2. In the ſame manner, the velocity in a circle at the diſtance SE would be found equal to the velocity that is acquired by falling through the radius ES with the gravity at E continued uniformly; whereas we found this velocity to be greater than the velocity in the circle in the ſubduplicate ratio of 2 to 1. The error ariſes from the ſuppoſing the gravity, when the body deſcends in the right line ES, or aQ, to be meaſured by the fluxion of the velocity of the body, or the ſecond fluxion of the ſpace deſcribed by it, and at the ſame time by the right line Tr: for theſe ſuppoſitions are inconſiſtent, becauſe it is $2Tr$, and not Tr, that meaſures the ſecond fluxion of the ſpace deſcribed by the body while it deſcends in the right line aQ with a motion uniformly accelerated, as has been ſhewn by ſeveral different methods in art. 74. 75. 97. & 254. The gravity is meaſured by the increment of the velocity that is generated in a given time when the gravity is invariable (and, conſequently, the velocity increaſes uniformly,) or by the difference of the ſpaces that would be deſcribed in the given time by the velocities at the beginning and end of that time continued uniformly: but Tr is equal to the half of this difference only, being equal to the exceſs of the ſpace that is deſcribed in the ſame given time when the motion is uniform-

ly

ly accelerated above the ſpace that would have been deſcribed if the motion had continued uniform from the beginning of that time. Therefore, in order to avoid ſuppoſitions that are inconſiſtent, QN which meaſures the centripetal force at Q, or E, muſt be ſuppoſed equal to $2Tr$, the velocity at E in the curve is to be repreſented by a mean proportional betwixt $\frac{1}{2}$QN and EB, or QN and E*b*, and the velocity in a circle at the diſtance SE by a mean proportional betwixt QN and the radius SE. The example we have deſcribed may ſerve to ſhew how miſtakes of this kind have ſometimes ariſen in the application of this method, and how they may be avoided.

441. When a circle is deſcribed by a centripetal force directed towards its center, the motion in the circle, the angular motion of the ray drawn from the center to the revolving body, the motion with which the area flows that is deſcribed by this ray, and the angular motion of the tangent at the body are all uniform. In other caſes, the area deſcribed by the ray SE drawn from the center of the forces S to the revolving body ſtill flows uniformly and meaſures the time; but the velocity at any point E is inverſely as SP the perpendicular from S on the tangent ET, the angular velocity of the ray SE about S is inverſely as the ſquare of SE, and the angular velocity of the tangent (or of the perpendicular SP about S) is inverſely as the rectangle contained by SP and the ray of curvature at E. Sir Isaac Newton's demonſtration of this may be repreſented
Fig. 193. in the following manner. When a body moves in a right line ET with an uniform motion, it deſcribes equal ſpaces on that line in any equal times; and if a ray be ſuppoſed to be drawn always from the body to a given point S that is not in that right line, this ray ſhall deſcribe equal areas about S in any equal times, Elem. 35. 1. A force that acts upon the body at any point E directed towards S has no effect on the magnitude of the area deſcribed in a given time by the right line drawn always from the body to S: it may accelerate or retard the motion of the body; but the area deſcribed about S is of the ſame magnitude as if no ſuch force had acted upon it. For, let ET be to EK as the velocity of the body in the direction ET is to the velocity that would be produced by the new impulſe

pulse in the direction ES, complete the parallelogram EKMT, join ST and SM, the body will now describe the diagonal EM in the same time that it would have described the side ET if no new force had acted upon it at E, and the triangle SEM is equal to SET. The same is to be said of any successive impulses provided they be always directed to the same center S; they never affect the fluxion of the area, which therefore flows in the same manner as if the body had proceeded in the tangent with the motion at E continued uniformly. It follows from this, FIG. 191.
that if AL be the tangent at A, and the triangle SAL be equal to the triangle SET, then AL and ET would be described in equal times by the motions at A and E continued uniformly. Therefore, if SX be perpendicular to AL in X, the velocity at A shall be to the velocity at E as AL is to ET, or as SP is to SX; that is, the velocity in any given figure is always inversely as the perpendicular from the center of the forces on the tangent.

442. This may be likewise demonstrated from art. 435. where it was shewn, that if the square of QR be equal to twice the area *a*QN*d*, so that QR may represent the velocity at E, the fluxion of QR shall be to the fluxion of *a*Q as QR is to E*b*. The fluxion of SP is to the fluxion of SE, or SQ, as SP is to E*b*, by art. 384 and the velocity with which SQ decreases is equal to the velocity with which *a*Q increases; therefore the velocity with which SP decreases is to that with which QR increases as SP is to QR; and the fluxion of the rectangle contained by QR and SP vanishes, by prop. 3. consequently, this rectangle is invariable, and QR which measures the velocity at E is inversely as the perpendicular SP. Because the area described about S flows uniformly and measures the time, the angular velocity of SE about S is reciprocally as the square of the distance SE, by art. 120. and if the sector ESI be equal to the FIG. 32.
area ESH, the sector ESI will be generated by SE if its an- n. 2.
gular motion about S be continued uniformly in the same time that the area ESH is described by the ray drawn from S to the body while it moves in the arch EH. The angular ve- FIG. 191.
locity of the tangent at E, or of the perpendicular SP, is to the angular velocity of SE as SE is to E*b*, and is inversely as the

the rectangle SE*b*, or the rectangle contained by SP and the ray of curvature EC.

443. The centripetal force at any point E is as the ſquare of the velocity at E directly, and E*b* half the chord of the circle of curvature that paſſes through S inverſely, by art. 422. Therefore this force is inverſely as the ſolid contained by E*b* and the ſquare of the perpendicular SP, or (becauſe E*b* is to SP as the fluxion of the diſtance SE to the fluxion of the perpendicular SP, by art. 384.) inverſely as a ſolid that is to the cube of SP as the fluxion of SE is to the fluxion of SP. It is likewiſe as the angular velocity of SP (or of the tangent at E) directly and a third proportional to SE and SP inverſely. The ſame force may be meaſured by the difference of the ſecond fluxion of SE conſidered as terminated at the tangent E*t* and its ſecond fluxion conſidered as terminated at the curve EH, or by the ſum or difference of the centrifugal force that ariſes from the motion of rotation (which is always inverſely as the cube of the diſtance SE, by art. 442. ſince the angular velocity of SE is reciprocally as the ſquare of SE) and the force by which the motion of
FIG. 166. L is accelerated or retarded at E in the right line ES, as in art. 431. and by ſeveral other theorems of this kind.

FIG. 191. 444. When a figure has the ſame curvature and the tangents inclined in the ſame angle to the rays drawn to the center of the forces at any two points, the centripetal forces at theſe points are reciprocally as the ſquares of the diſtances. For, ſince the chord of the circle of curvature is the ſame and SP is to SE in the ſame ratio in both caſes, the ſolid contained by EB and the ſquare of SP is as the ſquare of SE. Thus, the centripetal forces are reciprocally as the ſquares of the diſtances at the extremities of the ſame axis in any conic ſection, when the center of the forces is any where in that axis, or in a circle at the extremities of the diameter that paſſes through the center of the forces.

FIG. 194. 445. Let AEM be an ellipſe, and the centripetal force be directed to the center of the figure; then, ſince EB is equal to the parameter of the diameter that paſſes through E, (by art. 374.) the rectangle contained by the ſemidiameter OE and E*b* is equal to the ſquare of the ſemiconjugate OK. The rectangle

angle contained by OK and OP the perpendicular from O on the tangent is invariable. Therefore OE is inverſely as the ſolid contained by the ſquare of OP and E*b*; and the centripetal force is as the diſtance OE. This is likewiſe eaſily deduced from what was ſhewn in the introduction, p. 8. That, when the area EOM deſcribed about O (and, conſequently, the time in which the arch EM is deſcribed) is given, then TM is in an invariable ratio to the diſtance OE. The velocity at E is as the ſemidiameter OK conjugate to OE; becauſe the perpendicular from O on the tangent at E is inverſely as OK. The periodic time in the ellipſe is equal to the periodic time in a circle deſcribed at any diſtance OE by the centripetal force at E: For, if the tangent E*t* be ſuppoſed equal to the circumference of a circle whoſe radius is OK, the triangle EO*t* ſhall be equal to the area of the ellipſe, and the periodic time in the ellipſe equal to the time in which E*t* would be deſcribed by the motion at E continued uniformly, by art. 441. but the velocity at E is to the velocity in the circle at the ſame diſtance as OK is to OE, (by art. 425.) or as E*t* is to the circumference of the circle; conſequently, the time in E*t*, or the periodic time in the ellipſe, is equal to the periodic time in the circle. It follows, that the periodic times in all ellipſes are equal (as well as in circles, art. 432.) when the centripetal forces are as the diſtances. Let OE be equal to OQ, and the velocity at any point E equal to the velocity that would be acquired by falling from *a* to Q; then ſhall O*a* be equal to AH the diſtance betwixt the extremities of the tranſverſe and ſhorter axis. For, if *ad* and QN repreſent the centripetal forces at *a* and Q, as in art. 435. the trapezium *a*QN*d* ſhall be to the triangle OQN in the duplicate ratio of the velocity at E in the ellipſe to the velocity in a circle at the ſame diſtance, or as the ſquare of OK to the ſquare of OE; and the ſquare of O*a* to the ſquare of OE as the ſum of the ſquares of OK and OE to the ſquare of OE; conſequently, the ſquare of O*a* is equal to the ſum of the ſquares of OK and OE, or to the ſquare of AH. Hence, if a body be projected at A in a direction perpendicular to OA with the velocity that is acquired by falling from *a* to A, and a circle deſcribed from the center A with a radius equal to O*a* meet

OH perpendicular to OA in H, then OA and OH ſhall be the two ſemiaxes of the trajectory. In the ſame manner it is ſhewn, that when a centrifugal force directed from O is as the diſtance, an hyperbola is deſcribed that has its center in O: and as the time of the motion is meaſured by an elliptical or circular area when the centripetal force is as the diſtance, ſo it is meaſured by an hyperbolic area when the centrifugal force is as the diſtance; that is, by the meaſures of angles in the former caſe, and by the meaſures of ratios in the latter; which is agreeable to what was ſhewn above in art. 159. & 407.

FIG. 159. 446. Let the center of the forces be now in the focus S of any conic ſection; let the circle of curvature at E meet ES in B, and BX parallel to the tangent at E meet EX perpendicular to the ſame tangent in X; let XZ be perpendicular to EB in Z, and XZ ſhall be equal to the parameter of the tranſverſe axis of the figure, by the 8th property of the circle of curvature, in art. 375. Becauſe the triangles SEP, BEX are ſimilar, the ſquare of SP is to the ſquare of SE as the ſquare of EX to the ſquare of EB, or as EZ is to EB; conſequently, the ſolid contained by EB and the ſquare of SP is equal to the ſolid contained by EZ and the ſquare of SE. Therefore, ſince EZ is invariable in a given figure, the centripetal force towards S is reciprocally as the ſquare of the diſtance SE. It follows from what was ſhewn above (art. 424. & 427.) that the velocity in the parabola is to the velocity in a circle at the ſame diſtance as the ſquare root of 2 to unit, that the velocity in the ellipſe is to the velocity in ſuch a circle in a leſs ratio, and that the velocity in the hyperbola is to that velocity in a

FIG. 195. greater ratio. The time in which a revolution is completed in an ellipſe, is equal to the time in which a circle is deſcribed by the ſame centripetal force at a diſtance equal to half the tranſverſe axis; for the velocity at H the extremity of the ſhorter axis is equal to the velocity in this circle, by art. 426. But if HT be ſuppoſed to be taken upon the tangent at H equal to the circumference of that circle, the triangle HST ſhall be equal to the area of the ellipſe, and the periodic time in the ellipſe equal to the time in which HT would be deſcribed by the motion at H continued uniformly in the tangent, (by art. 441.) and, conſequently,

ſequently, equal to the periodic time in the circle deſcribed at the diſtance SH. Hence, when ellipſes are deſcribed by centripetal forces that are always inverſely as the ſquares of the diſtances from their common focus, the ſquares of the periodic times are as the cubes of the tranſverſe axes or of the mean diſtances. And ſince the tranſverſe axis is determined when the velocity and centripetal force at any given point in the curve are given, (by art. 427.) it follows, that all bodies projected from a given point in different directions, but with equal velocities, complete their revolutions and return to the ſame point again in equal times. The tranſverſe axis was determined in art. 427. The ſhorter axes of ſuch ellipſes are to each other as the perpendiculars from the center of the forces on the right lines in which the bodies are projected, or as the ſines of the angles contained by theſe right lines and the ray drawn to the center of the forces; becauſe the ſquare of SP is to the ſquare of SE Fig. 159. as EZ is to EB, and (ſince the tranſverſe axis with the diſtance SE and chord of curvature EB are given) the ſquare of SP is as the parameter EZ, or as the ſquare of the ſhorter axis. Let Fig. 195. SE be equal to SQ, and the velocity in the ellipſe at E be equal to the velocity that would be acquired by falling from *a* to Q, and S*a* ſhall be equal to the tranſverſe axis of the figure. For, let S*b* be taken upon S*a* equal to SH, and *bl* be the ordinate of the figure *ad*NQ at *b*; then, becauſe the velocity at H in the ellipſe is equal to the velocity in a circle at the ſame diſtance, the triangle S*bl* ſhall be equal to the area *abld*, (art. 436.) or to the difference of the rectangles S*bl* and S*ad*; conſequently, the triangle S*bl* is equal to the rectangle S*ad*, and 2S*a* is to S*b* as *bl* is to *ad*, that is, as the ſquare of S*a* is to the ſquare of S*b*; and S*a* is equal to 2S*b* or to the tranſverſe axis of the figure. Therefore, if a body was projected directly upwards from E in the right line SE with its motion at E, it would riſe to a diſtance from S equal to the tranſverſe axis of the figure; and the motion with which a circle is deſcribed would carry the body to a diſtance from S equal to the diameter of the circle. If a body begin to fall from *a* in the right line *a*S, it will be carried to an equal diſtance from S on the other ſide and will thence return again to *a* in the ſame time that a revolution is

completed in the ellipſe AEa whoſe tranſverſe axis is equal to S*a*, and the time in which it falls from *a* to S is one fourth part of this time, or of the periodic time in the ellipſe AaA, or of the time in which a circle is deſcribed at a diſtance from S equal to one half of S*a*. If a body be projected from A in a right line perpendicular to SA with the velocity that would be acquired by falling from *a* to A, then A ſhall be an apſis of the figure, and Sa the diſtance of the other apſis ſhall be equal to *a*A. When the velocity at E is to the velocity in a circle at the ſame diſtance as SE is to SP, the angular velocity of the ray SE is equal to the angular velocity of the ray in a circle deſcribed by the ſame centripetal force at the ſame diſtance: In this caſe, SE is equal to one half of the principal parameter of the figure, the tranſverſe axis is perpendicular to SE, the trajectory is an ellipſe, parabola or hyperbola, according as the angle SEP is greater, equal to, or leſs than half a right one; and the paracentric velocity in the trajectory is always greateſt at this point E. Becauſe the primary planets move in ellipſes that have their focus in the center of the ſun, ſo that the areas deſcribed by the ray drawn from any one planet to the ſun in equal times are equal, and the ſquares of the periodic times of the different planets are as the cubes of their mean diſtances from the ſun, (as was obſerved by the famous KEPLER,) and becauſe the ſame laws obtain in the motions of the ſatellites about the primary planets, Sir ISAAC NEWTON concludes, that there is a centripetal force extended over the ſyſtem to all diſtances directed towards the ſun that decreaſes in proportion as the ſquare of the diſtance from its center increaſes, and that a force decreaſing in the ſame manner is directed towards each body in the ſolar ſyſtem which at equal diſtances from their centers is as the matter in each of them. By comparing the effects of theſe forces with the deſcent of heavy bodies near the ſurface of the earth, he finds, that the power of gravity ſo well known to us is one inſtance of this general principle. The powers upon which the celeſtial motions depend being thus diſcovered, and reduced to known meaſures, he then proceeds to deduce theſe motions with their various inequalities from their principles; and a fuller account of this matter is expected from an author who has already

dy given ſeveral proofs of the great progreſs he has made in this moſt uſeful theory.

447. Some find it difficult to conceive how a body can revolve in an ellipſe, and, after approaching towards the center of the forces in deſcending from the higher apſis A to the lower apſis a, recede from it again by aſcending from the lower apſis a to the higher A, ſince the centripetal force is greater at a than at A. There are indeed various laws of the centripetal forces, which would either cauſe the body to deſcend from the apſis continually towards the center and at length fall into it, or to aſcend from the apſis continually and recede from the center for ever: but there are other laws which if the centripetal force obſerve, the body may approach to the center and recede from it by turns; and of this kind is the law which obtains in the ſolar ſyſtem. In order to diſtinguiſh thoſe caſes from each other, we are to obſerve, that when the velocity of the body at the higher apſis A is leſs than that which is requiſite to carry it in a circle about the center S at the ſame diſtance SA, the body muſt move in a curve that falls within that circle there, and muſt approach towards the center; while it deſcends, its velocity increaſes; and if its velocity increaſe in a higher proportion than that in which the velocities requiſite to carry bodies in circles about S increaſe, the velocity in the lower part of the curve may at length exceed the velocity in a circle at the ſame diſtance, and thereby become ſufficient to carry off the body again. In theſe caſes the velocity of the body in the orbit and the velocity in a circle deſcribed at the ſame diſtance may exceed each other thus by turns, the latter in the higher part and the former in the lower part of the orbit, and the body may approach towards the center and recede from it by turns. Thus, in the ſolar ſyſtem, the centripetal force increaſes as the ſquare of the diſtance from the center decreaſes, and the velocity that is requiſite to carry a body in a circle about the center increaſes only in the ſubduplicate ratio in which the diſtance decreaſes, art. 432. but the velocity in the orbit increaſes in a higher proportion while the diſtance decreaſes, (by art. 441.) ſo that though the former exceeds the latter at the higher apſis A, the latter by increaſing in a higher proportion becomes equal to it at

at the mean diſtance SH and exceeds it in its turn at the lower apſis a; and thus the body conſtantly revolves from one apſis to the other. This is further illuſtrated, by comparing the centripetal force in the orbit with the centrifugal force that ariſes from the circular motion of the body about S: for this centrifugal force increaſes always in proportion as the cube of the diſtance from the center decreaſes, and conſequently in a higher proportion than the centripetal force; ſo that each of theſe forces prevails in its turn, the centripetal force in the higher part, and the centrifugal force in the lower part of the orbit.

448. In general, a body may approach towards the center S and recede from it by turns when the velocities that are requiſite to carry bodies in circles about S increaſe in a leſs proportion than that in which the diſtances decreaſe. But if the centripetal force be ſuch, that theſe velocities increaſe in the ſame proportion in which the diſtances decreaſe, a body cannot revolve about the center S in this manner; if it begin to approach towards the center S when it proceeds from the apſis A, it will approach continually to the center till it fall into it; or if it begin to recede from S when it ſets out from the apſis a, it muſt recede from the center to greater and greater diſtances for ever. For, ſhould we ſuppoſe the body to deſcend in this caſe from the higher apſis A to the lower apſis a, ſince the velocity in the orbit at A would be to the velocity at a as Sa is to SA, (by art. 441.) that is, (by the ſuppoſition,) as the velocity in the circle at the diſtance SA is to the velocity in the circle at the diſtance Sa, it follows, that the velocity in the orbit at A would be to the velocity in a circle at the ſame diſtance as the velocity in the orbit at a is to the velocity in a circle at the diſtance Sa; and that, ſince the velocity in the orbit at A is leſs than the velocity in a circle at the diſtance SA, the velocity in the orbit at a would likewiſe be leſs than the velocity in a circle at the diſtance Sa; ſo that the body would continue to approach to the center after it comes to a. In the ſame manner, if the velocity at a in the orbit exceed the velocity in a circle at the diſtance Sa, the velocity in the orbit at A would likewiſe exceed the velocity in a circle at the diſtance SA, and the body would continue to recede from S after it comes to A.

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When the centripetal force is reciprocally as the cube of the diſtance from S, the velocities that are requiſite to carry bodies in circles about S increaſe in the ſame proportion that the diſtances from S decreaſe, by art. 432. therefore the body after it proceeds from the apſis (unleſs it move in a circle) muſt approach continually to the center or recede from it for ever, and the figure can have no more than one apſis. When the centripetal force is reciprocally as any higher power of the diſtance, the velocities that are requiſite to carry bodies in circles about S increaſe in a higher proportion than that in which the diſtances decreaſe; conſequently, the body after it ſets out from the apſis either approaches continually to the center till it fall into it, or recedes from it for ever. Thus it was ſhewn in art. 440. that, when the centripetal force is reciprocally as a power of the diſtance whoſe exponent is any number that exceeds 3 by any fraction $\frac{1}{s}$, if it be projected at A with a velocity that is to the velocity in a circle at the diſtance SA in the ſubduplicate ratio of $2s$ to $2s+1$, it will fall into the center in a number of revolutions denoted by $\frac{1}{2}s$. But when the centripetal force is reciprocally as a power of the diſtance whoſe exponent is leſs than 3, the velocity in the trajectory increaſes while the diſtance decreaſes in a higher proportion than the velocity in circles deſcribed by the ſame centripetal forces, by art. 441. and in thoſe caſes the body may approach to the center and recede from it by turns.

449. In general, the velocity in any orbit becomes equal to the velocity in a circle deſcribed at the ſame diſtance by the ſame centripetal force when the angle ESP is a *maximum*, or when the angle contained by the tangent and ray drawn to the center of the forces is a *minimum*, by art. 424. or ſuppoſing (as in art. 435.) that the centripetal force at any diſtance SQ (or SE) is meaſured by the ordinate QN, and that the velocity at A is that which would be acquired by falling from a to A, when the area aQNd becomes equal to the triangle SQN; that is, if the centripetal force be reciprocally as the power of the diſtance whoſe exponent is m, when the rectangle SQN is to the rectangle Sad as 2 is to $3-m$, or when the power of Sa whoſe expo- FIG. 191.

exponent is $m - 1$ is to the ſame power of SE in that ratio. Let a be the other apſis of the trajectory, and, SG being made equal to Sa, let GF repreſent the centripetal force at the diſtance SG, (or Sa,) and SA ſhall be to SG, in general, in the ſubduplicate ratio of the area *a*GF*d* to the area *a*AD*d*, (by art. 435. & 442.) that is, in the preſent ſuppoſition, in the ſubduplicate ratio of the difference betwixt the rectangles SGF and S*ad* to the difference of the rectangles SAD and S*ad*. The angular velocity of the ray SE in the trajectory becomes equal to the angular velocity in a circle deſcribed by the ſame centripetal force at the ſame diſtance, and the paracentric velocity is greateſt, in general, when the velocity in the curve is to the velocity in a circle at the ſame diſtance as SE is to SP, or when SE is to SA in the ſubduplicate ratio of the area *a*AD*d* to

Fig. 196. the triangle SQN. Thus, for example, if the centripetal force be the ſame in all diſtances, the velocity in the trajectory becomes equal to the velocity in a circle when SE is two thirds of S*a*; and Sa (or SG) is to SA in the ſubduplicate ratio of *a*G to *a*A, from which it follows, that, if *a*A be biſected in K and a circle deſcribed through K from the center S meet AL perpendicular to SA in L, then Sa ſhall be equal to the ſum of KA and AL; and A is the higher or lower apſis according as SA is greater or leſs than this ſum; but when SA is equal to this ſum, a circle is deſcribed about the center S: When the cube of SE is double of the ſolid contained by *a*A and the ſquare of SA, the angular velocity of the ray SE is equal to the angular velocity of a ray in a circle that is deſcribed at the ſame diſtance by the ſame centripetal force, and the paracentric velocity is then greateſt at E.

Fig. 191. 450. In the logarithmic ſpiral, SP is to SE in an invariable ratio, E*b* is equal to SE, and the centripetal force towards S is inverſely as the cube of SE, by art. 443. In the reciprocal or hyperbolic ſpiral (deſcribed in art. 344.) ST is invariable; from which it follows, that the fluxion of SP is to the fluxion of SE as the cube of ST to the cube of ET, or the cube of SP to the cube of SE, and that SP is to E*b* in the ſame ratio; conſequently, the centripetal force towards S is inverſely as the cube of the diſtance SE. The centripetal forces obſerve the

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ſame law when the ſquare of SE is to the ſquare of SP as the ſum or difference of any given ſpace and the ſquare of SE is to a given ſquare. The conſtruction of the figures which have this property is given, *Harmon. menſur. p.* 31. and *Philoſ. Tranſ. n.* 317. When any figure conſtructed in art. 392. is deſcribed by a centripetal force directed towards S, this force is inverſely as the power of the diſtance SL whoſe exponent is $3 - \frac{2}{n}$, becauſe LI is to SL as n is to $n - 1$, SP to SL as SA is to SM, and, conſequently, SP always as the power of SL whoſe exponent is $1 - \frac{1}{n}$. When any of the figures conſtructed in art. 393. are deſcribed by a force directed towards S, SP is as the power of SL whoſe exponent is $1 + \frac{1}{n}$, LI is to SL in an invariable ratio, and the force is reciprocally as the power of SL whoſe exponent is $3 + \frac{2}{n}$: And this is the converſe of what was ſhewn above in art. 436. & 437. FIG. 171. FIG. 172.

451. Let AM be any curve line that can be deſcribed by a force that is as the power of the diſtance whoſe exponent is any number m, let the angle ASL be to ASM as $m + 3$ is to 2, and SL be to SA as the power of SM whoſe exponent is one half of $m + 3$ is to the ſame power of SA; then the curve AL may be deſcribed by a centripetal force directed towards S, that is, as the power of the diſtance SL whoſe exponent is $\frac{4}{m+3} - 3$. The demonſtration may be deduced from art. 394, but will appear more eaſily afterwards. Thus, if m be ſuppoſed ſucceſſively equal to 2, 1, $\frac{1}{2}$, 0, $-\frac{1}{2}$, $-\frac{3}{5}$, the curve AL may be deſcribed by a centripetal force directed towards S that is inverſely as the power of the diſtance whoſe exponent is $2\frac{1}{5}$, 2, $1\frac{6}{7}$, $1\frac{2}{3}$, $1\frac{3}{5}$, $1\frac{1}{3}$, reſpectively. The point A is an apſis common to AM and AL. If B be the other apſis of AM and D the other apſis of AL, the angle ASD muſt be to ASB as $m + 3$ is to 2, by the conſtruction: Therefore, if $M - 3$ be ſuppoſed equal to m, and $N - 3$ equal to $\frac{4}{m+3} - 3$, the angle ASD FIG. 173.

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shall be to ASB in the subduplicate ratio of M to N. And this is agreeable to what is shewn by Sir Isaac Newton of the motion of the apsides (*Princip. lib.* 1. *prop.* 45. *ex.* 2.) when the excentricity of the orbit is supposed incomparably little, which case only he has considered.

Fig. 196. 452. Suppose now that the revolving body describes any trajectory AEM in a medium that resists its motion. Let the centripetal force at E by which the curve would be described in a void be to the centripetal force at E by which it is described in the medium as any given invariable right line S*a* is to SZ, then the resistance at E shall be inversely as a space that is to the square of SP the perpendicular from S on the tangent as the fluxion of the curve AE is to the fluxion of SZ; and the density of the medium (supposing the resistance to be as the density of the medium and square of the velocity together) shall be reciprocally as a right line that is to SZ in the same ratio. For, let the velocity at E in the void be represented by E*l* and the centripetal force by E*k*, the velocity at E in the medium by EL and the centripetal force by EK, the resistance by ER, and, the fluxion of the curve being represented by EN, let NV be perpendicular to the tangent in V that EV may represent the fluxion of the ray SE. When the direction of the motion at E is in the right line ET that forms an acute angle with the ray SE, the velocity is accelerated by a force that is to the centripetal force EK as EV is to EN the fluxion of the curve, and is at the same time retarded by the resistance ER; the rectangle contained by 2EN and the difference of these two forces (that is, the excess of the rectangle 2KEV above the rectangle 2NER) measures the fluxion of the square of EL, by art. 424. and the fluxion of the solid contained by S*a* and the square of EL is measured by the solid contained by S*a* and 2KEV—2NER. But the square of EL is to the square of E*l* as EK is to E*k* (by art. 429.) or as SZ is to S*a*; and the solid contained by S*a* and the square of EL is always equal to the solid contained by SZ and the square of E*l*; consequently, their fluxions are equal, and the solid contained by S*a* and 2KEV — 2NER is equal to the fluxion of the solid contained by SZ and the square of E*l*: but this fluxion consists of two parts, the solid contain-

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ed by SZ and the ſpace which meaſures the fluxion of the ſquare of E*l*, (that is, the ſolid contained by SZ, 2E*k* and EV, by art. 435. or the ſolid contained by S*a*, 2EK and EV, ſince SZ is to S*a* as EK is to E*k*,) and the ſolid contained by the ſquare of E*l* and the fluxion of SZ. From which it appears, that this laſt ſolid is equal to the ſolid contained by S*a*, 2NE and ER; but that the fluxion of SZ is negative, or that SZ muſt decreaſe while the body approaches to S in the arch EM and the ray SE decreaſes. Therefore, if Z*z* meaſure the fluxion of SZ, the rectangle contained by 2S*a* and EN ſhall be to the ſquare of E*l* as Z*z* is to ER that meaſures the reſiſtance at E; and, becauſe E*l* that meaſures the velocity in a void at E is always inverſely as SP, it follows, that the reſiſtance at E is directly as Z*z* the right line which meaſures the fluxion of SZ and inverſely as the ſolid contained by the ſquare of SP and EN the right line that meaſures the fluxion of the curve AE. When the direction of the motion at E is in the right line E*t* that forms an obtuſe angle with the ray SE, the velocity is retarded by a force that is to the centripetal force EK as EV is to EN, and at the ſame time by the reſiſtance ER; and it will appear in the ſame manner, that the reſiſtance ER is directly as Z*z* and inverſely as the ſolid contained by the ſquare of SP and EN, but that in this caſe SZ muſt decreaſe while the ray SE increaſes and the body recedes from the center. Nor can a body move in the trajectory, either by proceeding from E towards M or towards A, unleſs SZ begin to decreaſe from the moment when the body ſets out from E. If the area ASE be ſuppoſed to flow uniformly, the fluxion of the curve AE ſhall coincide with the velocity by which it is deſcribed in a void (by art. 441.) ſo that we may ſuppoſe E*l* equal to EN; and 2S*a* ſhall be to EN as Z*z* is to ER, or the reſiſtance ſhall be always as the rectangle contained by the right lines that meaſure the fluxions of the curve AE and of SZ; and this coincides with prop. 23. par. 2. *Deſcript. curv.*

453. Becauſe ER is to Z*z* as the ſquare of E*l* is to the rectangle contained by 2S*a* and EN, and the ſquare of EL is to the ſquare of E*l* as SZ is to S*a*, ER is to Z*z* as the ſquare of EL to the rectangle contained by SZ and EN. Therefore the

 denſity

density (which is as ER directly and the square of EL inversely) is as Zz directly and the rectangle contained by SZ and EN inversely; consequently, if the curve AEM be extended into a right line and the ordinate at E be always equal to SZ, the density at E shall be always inversely as the subtangent of this figure.

454. Let AEM be any trajectory that can be described in a void by a centripetal force that is inversely as any power of the distance SE, and let it be described in a medium by a centripetal force that is likewise inversely as some power of the distance SE; let ST perpendicular to SE meet the tangent in T, and the density at E shall be always inversely as the tangent ET. For let Ek and EK be inversely as the powers of the distance whose exponents are n and m respectively; then SZ shall be always as the power of the distance whose exponent is $n - m$, and Zz the fluxion of SZ shall be to the fluxion of SE in the ratio compounded of that of SZ to SE and that of $n - m$ to unit, by art. 167. therefore the density shall be inversely as a right line that is to SE as the fluxion of the curve AE is to the fluxion of the ray SE, that is inversely as the tangent ET.

455. The resistance at E is to the centripetal force in the medium at E as the rectangle contained by one half of Eb and Zz is to the rectangle contained by EN and SZ, or (because Eb is to SP as the fluxion of SE to the fluxion of SP, by art. 384.) in the ratio compounded of the ratios of SP to 2SZ, of PE to SE and of Zz to the fluxion of SP. For example, if the trajectory be the logarithmic spiral, and EK the centripetal force in the medium be inversely as the power of the distance SE whose exponent is m; then, since Ek is inversely as the cube of the distance SE, SZ must be as the power of SP whose exponent is $3 - m$, the fluxion of SZ must be to the fluxion of SP in the ratio compounded of that of SZ to SP and that of $3 - m$ to unit, by art. 167. and the resistance to the centripetal force in the ratio compounded of that of $\frac{1}{2}$PE to SE and that of $3 - m$ to unit. The body cannot descend towards the center S in this spiral unless 3 be greater than m; and if it ascend in the spiral, m must be greater than 3, because SZ must decrease in both cases.

cases, and because SE is to SP in an invariable ratio. The density of the medium is inversely as ET or the distance from S; as is shewn by Sir ISAAC NEWTON, *Princip. lib.* 2. *prop.* 15. and by Mr. BERNOULLI, *Mem. de l'Acad. Royale des Sciences* 1711. If we suppose the trajectory to be one of those con- FIG. 170. structed in art. 392. or 393, and r be supposed equal to $3 - \frac{2}{n}$. & 171. in the former, or to $3 + \frac{2}{n}$ in the latter, SZ will be as the power of the distance whose exponent is $r - m$ (by art. 452.) and SP as the power of the distance whose exponent is one half of $r - 1$. Therefore (by art. 167.) the resistance at any point L shall be to the centripetal force at L in the ratio compounded of that of LP to SL and that of $r - m$ to $r - 1$. The body cannot approach to S in any of those figures unless r be greater than m, and it cannot recede from S in any of them unless r be less than m. The resistance and density vanish at the apsis A in these figures, or in any of those that can be described in a void by a force that is as any power of the distance. It was shewn above, (art. 429.) that the ratio of the velocity in the curve to the velocity in a circle described in a void at the same distance by the same centripetal force is always the same as when the curve is described in a void.

456. If the centripetal force act in parallel lines, let AD per- FIG. 197. pendicular to those lines flow uniformly; and let DZ be now to a given right line Da as EK, which measures the centripetal force in the medium at E, is to Ek that measures the centripetal force in the void at E or the second fluxion of the ordinate DE (by art. 418.) and, the rest remaining, the resistance ER will be to Zz as EN is to $2Da$. If the centripetal force in the medium be supposed uniform, the rectangle contained by DZ and Ek must be invariable, and the fluxion of DZ to the fluxion of Ek (or the third fluxion of the ordinate DE) as DZ is to Ek, (by prop. 3.) or as the rectangle contained by EK and Da is to the square of Ek. Therefore the resistance ER is to the gravity EK as the rectangle contained by the right lines that measure the fluxion of the curve AE and the third fluxion of the ordinate DE is to twice the square of Ek that measures the

the second fluxion of the same ordinate. The velocity at E in the medium is to the velocity at E in the void in the subduplicate ratio of EK to E*k*; and the density of the medium at E is reciprocally as a right line that is to E*k* as the fluxion of the curve is to the fluxion of E*k*, or as the right line that measures the third fluxion of the ordinate DE directly and the rectangle contained by the right lines that measure the fluxion of the curve and the second fluxion of DE inversely. By these theorems the resistance and density of the medium may be computed when the nature of the curve is known; but they may be represented geometrically in the following general manner.

Fig. 149. & 150. 457. Suppose, as in art. 366. that the rectangle MTK is always equal to the square of ET, that BV the tangent of the curve BKF at B meets ET the tangent of EM in V; then, if the curve be described by an uniform gravity that acts always in lines parallel to EB, in a medium whose resistance is as its density and the square of the velocity of the body together, the resistance at E shall be to the gravity as 3EB is to 4EV, and the density of the medium shall be always inversely as the tangent EV. But if the resistance be supposed to be as the density and velocity together, let the angle EV*u* be made equal to EBV on the same side of EV, and V*u* meet EB in *u*; then the density shall be inversely as $\sqrt{Eu}$. Let the figure, for example, be any conic section, O the center, EG a chord in the direction of the gravity; let the angle EO*k* be made equal to GET and O*k* meet EG in *k*, and the resistance shall be to the gravity as 3O*k* is to 2OE; and if the tangent at E meet the semidiameter that bisects EG in V, the density shall be inversely as EV: for, since the triangles EV*b*, OE*k* are similar (by the 5th property of the circle of curvature, art. 375.) E*b* is to EV as O*k* to OE. When the section is a circle, O*k* becomes perpendicular to E*b*. When the section is an hyperbola and EB is parallel to one of the asymptotes, the tangents BV and ET intersect each other in that asymptote at V, and the resistance is to the gravity as 3EV is to 2AV. When the tangent in any figure becomes perpendicular to EB the direction in which the gravity acts, the ratio of the resistance to half the gravity is the same ratio which Sir Isaac Newton calls the index of the variation of

Fig. 158. Fig. 154.

of curvature, or the ratio of the fluxion of the ray of curvature to the fluxion of the curve. In computing the resistance of the medium from the second and third fluxions of the ordinate, or the right lines that represent them, regard must be had to what was shewn of these fluxions in chap. 4. and art. 255. in order to avoid mistakes like to those described in art. 440.

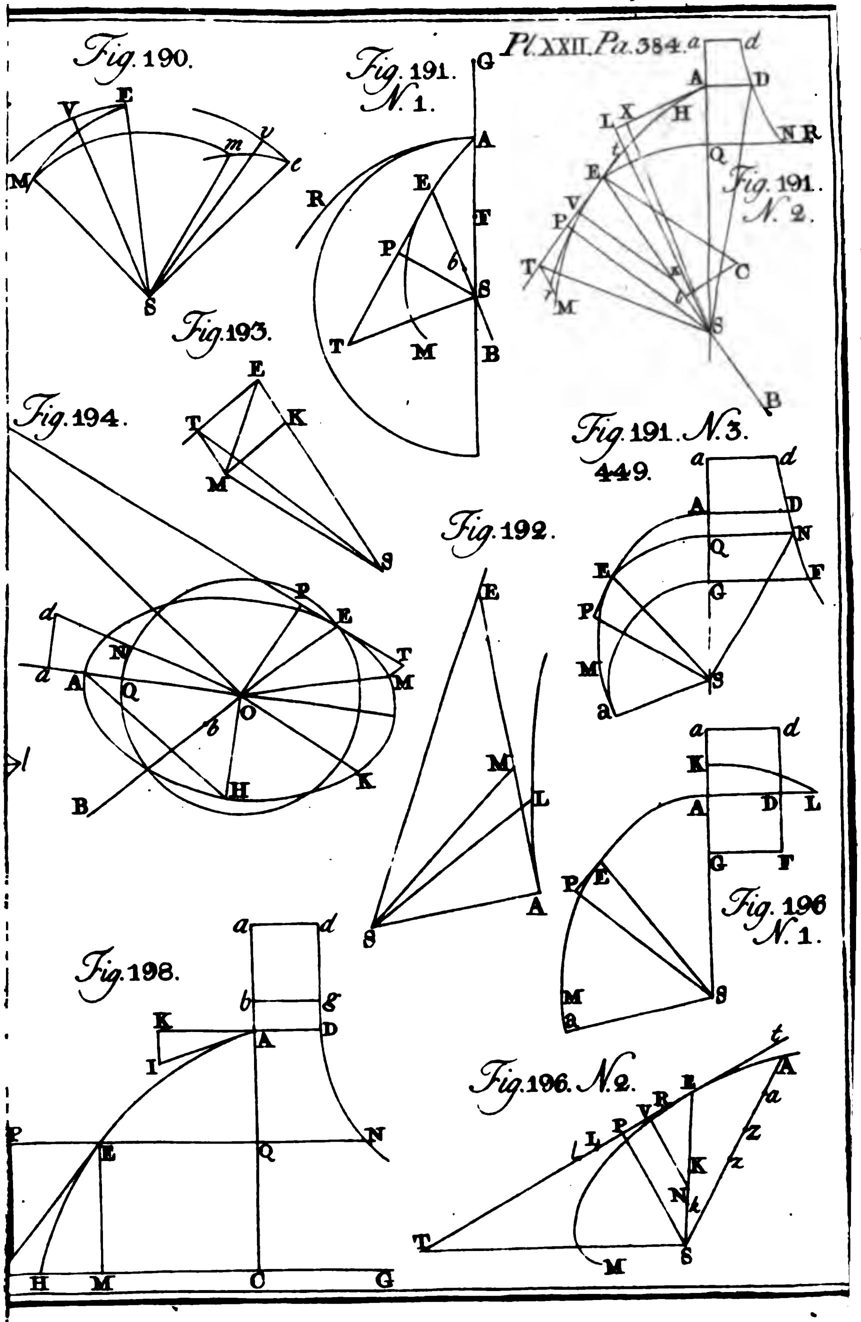
458. When the gravity acts in parallel lines, and is either uniform or varies as any power of the distance from a given plane GH, and AEH is any trajectory that could be described in a void by a force that is also as any power of the distance from GH, the density of the medium at E is always inversely as the tangent ET terminated by the point of contact E and by GH in T. The demonstration is similar to that of art. 454. FIG. 198.

459. Let AEH be any trajectory described in a void, or in a medium that has no resistance, by a force that acts always in right lines (as EM) perpendicular to GH and in the plane of the trajectory. Upon EQ that is parallel to GH take QN so as always to represent the force at the distance CQ (or EM) from GH; let AD represent the force at the distance CA, and the velocity at A be such as would be acquired by falling from *a* to A with an uniform gravity equal to AD; complete the rectangle AD*da*; and the velocity at E shall be to the velocity at A in the subduplicate ratio of the area *a*QND*d* to the rectangle *a*D. Let AK and EP parallel to GH and equal to each other represent the constant fluxion of QE; let KI and PR meet the tangents AI and ER in I and R; let *ab* be to *a*A as the square of AK to the square of AI, and *bg* parallel to AD meet D*d* in *g*: then ER shall be to AI as the velocity at E to the velocity at A, or in the subduplicate ratio of *a*QN*d* to *a*D; consequently, PR is to EP, or the fluxion of the ordinate EM to the fluxion of the base CM, in the subduplicate ratio of the area *b*QND*g* to the rectangle *ag*; and hence the construction of these trajectories may be deduced when the law of the force is given. For example, when the force is inversely as the cube of the distance from GH, the trajectory is a conic section. When the force is as the distance from GH, let A be the apsis of the trajectory; take QE always in the same ratio to the arch AL whose cosine is CQ (CA being radius) as *a*A is to FIG. 199.
one

one half of CA; then E ſhall be in the trajectory, and the body will move from A to the right line CH always in the
FIG. 198. ſame time. When the force is inverſely as the ſquare of the diſtance from GH, if C*b* be equal to 2CA, the trajectory is a ſemicubical parabola that has its cuſpis at H ſo that CH is to a mean proportional betwixt CA and *ab* as 2 is to 3; and the cube of EM is equal to the ſolid contained by the ſquare of HM and a third proportional to *ab* and $\frac{3}{2}$ CA. It is conſtructed by the area of the circle when C*b* is leſs than 2CA, and by an hyperbolic area when C*b* is greater than 2CA. In general, QE is always as the time in which a body would deſcribe AQ by falling from A to Q by the ſame forces directed towards C.

FIG. 198. 460. The ſine of the angle MER is to the ſine of CAI in the ſame ratio, when the velocity at A with the diſtance AQ of the parallels AK and QE is given, and the force perpendicular to GH or QE is the ſame at the ſame diſtance from QE. For AI is to AK as the radius to the ſine of the angle AIK, or CAI, and ER is to EP as the radius to the ſine of the angle MER; conſequently, the ſine of the angle MER is to the ſine of CAI as AI to ER, or in the ſubduplicate ratio of *a*D to the area *a*QN*d*; but this ratio remains the ſame in all the different poſitions of the tangent AI: therefore, when a ray of light paſſes through any given medium that acts upon it in parallel lines perpendicular to the planes that terminate the medium, the ſine of the angle of refraction is to the ſine of the angle of incidence in an invariable ratio, and the velocity with which the ray emerges at E is to the velocity of its incidence at A in the inverſe ratio; as Sir ISAAC NEWTON has ſhewn in a different man-
FIG. 200. ner, *prop.* 94. & 95. *lib.* I. *Princip.* If we ſuppoſe the ray to move from E towards A, and the angle MER to be increaſed till its ſine be to the radius in the ſubduplicate ratio of *a*D to *a*QN*d*, the angle CAI will become a right one, or A will be the apſis of the trajectory, and the ray will be reflected from A ſo as, after returning to the right line EQ*e* at *e*, to emerge in an angle equal to the angle of incidence MER.

461. When the body is projected from A in a right line AI that is not in the plane ACH, the force being ſtill perpendicular


Pl. XXII. Pa. 384.
Fig. 190.
Fig. 191. N. 1.
Fig. 191. N. 2.
Fig. 193.
Fig. 194.
Fig. 191. N. 3.
449.
Fig. 192.
Fig. 196 N. 1.
Fig. 198.
Fig. 196. N. 2.

cular to CH, let AI repreſent the velocity of the projection, and IV parallel to CH meet a plane AKB that is perpendicular to CH in V; and the ſame trajectory will be deſcribed as when the body is projected in the plane AKB with the velocity AV, and gravitates towards C by a force that varies in the ſame manner as the force towards the axis CH, while the plane AKBC is at the ſame time carried along CH with the velocity IV.

462. Hitherto we have ſuppoſed the forces to act in right lines that meet in one given point, or are perpendicular to one given right line. Let us now ſuppoſe that centripetal forces are directed towards any number of fixed points, and that the force towards each is always the ſame at the ſame diſtance from it; then if a body move from A to E, the velocity at E ſhall be the ſame whether it move in any curve line AZE or in the chord AE, if the velocity at A be the ſame in both caſes. For by what was ſhewn in art. 435. the increment or decrement of the velocity generated by the force directed towards any one center C is the ſame whether the body move in a curve or in a right line from A to E, if it ſet out from A with the ſame velocity; and when any number of forces directed towards ſeveral centers act upon the body, the ſquare of the velocity at E is meaſured by the aggregate of the ſame areas, whether the body move in the curve AZE or chord AE. Fig. 201.

463. Let AZE be ſuch a ſurface that when a body is placed upon it the actions of all the centers towards which it is attracted ballance each other, and the body is ſuſtained by the ſurface ſo as to move neither way; then if a body attracted towards thoſe centers in the ſame manner move in any line that meets this ſurface in A and E, its velocities at theſe points ſhall be equal; and if a body move from any given point with a given velocity and arrive at this ſurface, its velocity when it comes to this ſurface will be always the ſame. Hence the proper reſolution is deduced of a project for a perpetual movement mentioned by a celebrated Author. A loadſtone at A is ſuppoſed to have a ſufficient force to bring up a heavy body along the plane FA from F to B, whence the body is ſuppoſed to deſcend by its gravity along the curve BEF till it return to its firſt place F, and thus to riſe along the plane FA and deſcend along the curve BEF Fig. 202.

continually. But ſuppoſing BZE to be the ſurface upon which if a body was placed the attraction of the loadſtone and the gravity of the body would ballance each other, this ſurface ſhall meet BEF at ſome point E betwixt A and F, and the body muſt ſtop in deſcending along AEF at the point E. See WILKINS *Mathem. mag. Book* II. *Chap.* 13.

FIG. 201. 464. Let AZE be a trajectory deſcribed by forces directed towards the centers C and S, the ſquare of the velocity at A be repreſented by the rectangle *a*D, and the forces at A towards C and S by the ordinates AD and A*d* perpendicular to CA and SA, reſpectively; let CQ and S*q* be taken upon CA and SA reſpectively equal to CE and SE, and the ordinates QN, *qn* always repreſent the forces towards C and S at the Diſtances CE and SE; then if the ſquare of V be equal to twice the aggregate of the areas *a*D, AQND and A*qnd* joined with their proper ſigns, V ſhall meaſure the velocity at E. And in the ſame manner V is determined when there are more centers that attract the body. Let EH be taken upon EC, and EL upon ES reſpectively equal to QN and *qn*, complete the parallelogram EHKL, then the force at E that reſults from the attraction of the ſeveral centers ſhall act in the direction EK and be meaſured by EK. Let the circle of curvature at E meet EK in B, biſect EB in *b*, and the rectangle *b*EK ſhall be equal to the ſquare of VV, by art. 440. And if KR parallel to the tangent at E meet any other chord of the circle of curvature, as ED, in R, and ED be biſected in *d*, the rectangle *d*ER ſhall be equal to the ſquare of V; for the angle ERK is equal to EBD, EB to ED as ER to EK, and the rectangle *d*ER equal to *b*EK. Hence, if KP be perpendicular to the tangent at E in P, the rectangle contained by KP and the ray of curvature is equal to the ſquare of V; and if KR meet TE (that is drawn from E to any fixed point T) in *x*, the rectangle contained by E*x* and half the chord of curvature that paſſes through T ſhall be equal to the ſquare of V.

FIG. 203. 465. Let T be any fixed point, TC the baſe of the figure, and TA perpendicular to TC; let EM the ordinate from E meet the baſe in M and EN parallel to the baſe meet TA in N; let KH parallel to TA meet EM in H; then the time being always

ways supposed to flow uniformly, and the fluxion of the curve being represented by V which measures the velocity at E, the second fluxion of the ordinate shall be always measured by KH, and the second fluxion of the base by EH, or KL: That is, if while E describes the trajectory, EM and EN be always perpendicular to TC and TA in M and N, the powers by which the motions of the points M and N are accelerated or retarded shall be measured by EL and EH. For let E*t* be taken upon the tangent equal to V, and *t*I parallel to TA meet EN in I, that E*t*, EI and I*t* may measure the fluxions of the curve base and ordinate, respectively. Let *tt* measure the second fluxion of the ordinate, and supposing O to be the center of curvature, let Om be perpendicular to EM in m; and KP being perpendicular to E*t*, let P*z* be perpendicular to EN in *z*. Then the second fluxion of the curve (or the power that accelerates the motion of E) shall be measured by EP. The rectangle contained by OE and I*t* is equal to that which is contained by Om and E*t*, and (because the fluxions of those rectangles are equal, and while CE increases Om and I*t* decrease) the rectangle contained by EI and E*t* is equal to the sum of the two rectangles contained by Om and EP, and by OE and *tt*. By the last article KP is to E*t* (or V) as E*t* is to OE; and the sum of KH and P*z* is to KP as EI to E*t*; therefore the Sum of the two rectangles contained by OE and KH, and by OE and P*z*, is equal to the rectangle contained by EI and E*t*, or to the sum of the two rectangles contained by OE and *tt* and by Om and EP; but (because of the similar triangles OmE and P*z*E) the rectangle contained by OE and P*z* is equal to that which is contained by Om and EP; consequently, the rectangle contained by OE and KH is equal to the rectangle contained by OE and *tt*, and KH is equal to *tt* which was supposed to measure the second fluxion of the ordinate EM. In the same manner it appears that KL, or EH, measures the second fluxion of the base TM. This theorem holds, not only when the angle MTN is right, but when MTN, TME and TNE are any given angles, providing KL parallel to TC meet EM in L, and KH parallel to TA meet EN in H; for KH will always measure the second fluxion of TN, and KL the second fluxion of TM.

 466. This

466. This principle however, that when the force which results from the attraction of all the centers is measured by EK, and is resolved into the forces EH and EL parallel to TA and TC respectively, the second fluxions of the ordinate EM and base TM are measured by EH and EL respectively, may be admitted without this Proof. For if we resolve the motion at E in the direction E*t* into a motion EI in a right line parallel to TC, and a motion E*i* parallel to TA, we may conceive the body to descend in the right line EN with a velocity EI that is accelerated by the force EH, while the right line EN moves parallel to itself along TA with a velocity E*i* that is retarded by the force EL; and thus it may appear (as in art. 418.) that the force EH measures the second fluxion of EN, and the force EL the second fluxion of EM. The position of EH and EL serve likewise to shew whether these second fluxions are to be considered as affirmative or negative: thus when TM decreases and EM increases, and EK is within the angle MEN, the second fluxions of TM and EM are to be both considered as negative; but if EK be within the angle *i*EN, the second fluxion of EM is to be considered as positive.

467. This principle being admitted, several of the theorems in the preceeding articles may be readily demonstrated from it. Thus if the force EK be directed towards any fixed point that is any where in the right line EM, EK will coincide with EL, so that EH will vanish; consequently the centripetal force will be measured by the second fluxion of the ordinate EM when TM flows uniformly, as in art. 423. When the force is directed towards the fixed point T, the area described about T flows uniformly, or the fluxion of the area CTE is invariable; for the area CTE is half the sum of the areas CME and CTNE, the fluxions of which are measured by the rectangles IM and *i*N, by prop. 4. The fluxion of the rectangle IM is measured by the sum of the rectangles HM and I*i*, and the fluxion of *i*N by the sum of the rectangles LN and I*i*, by prop. 3. But while CM and ME increase, EN and I*t* decrease; consequently the second fluxion of the area CTE (or the fluxion of the sum of the rectangles IM and *i*N) is measured by the excess of HM and I*i* above LN and I*i*, that is, by the excess of HM above LN, or of KM above KN.

But

But when EK passes through T, this excess vanishes, and the area described about T flows uniformly; which is the first proposition of the first book of Sir ISAAC NEWTON's principles, and was demonstrated after his method above in art. 441. and in a manner that differs little from Mr. HERMAN's in art. 442.

468. In general it appears that the second fluxion of the area CTE is measured by the excess of the rectangle KM above KN, or (if EK meet TA in V) by the rectangle contained by KL and TV; and the fluxion of the area CTE increases or decreases according as EK is on the side of ET towards which the body moves, or on the opposite side. The theorem in art. 431. may be likewise made more general; for let E*u* be a circle described about the center T, and the uniform angular velocity of T*u* be equal to the angular velocity of TE while E moves in the trajectory. Let *up* parallel to TE meet the trajectory in *x*, and meet T*p* perpendicular to TE in *p*; and let *e* move in the right line TE so that T*e* be always equal to T*x*: then (by art. 385.) the second fluxion of T*e* (or T*x*) shall be equal to the difference of the second fluxions of *px* and *pu* when *x* and *u* set out together from E. But if K*y* be perpendicular to TE in *y*, E*y* shall measure the second fluxion of *px* at that term; and the force by which the circle E*u* is described about T, is measured by the second fluxion of *pu* at the same term, by art. 422. From which it follows, that the force by which the motion of *e* is accelerated or retarded, when it sets out from E, is equal to the difference of the force E*y* and the force by which the circle E*u* is described about T, or the centrifugal force that arises from the circulatory motion of E about T. Of this principle, see *the Laws of the Moon's motion according to gravity*, p. 65 & 66. FIG. 203. n. 2.

469. We supposed the trajectory described by E to be in one plane in art. 466 & 467. But similar conclusions are easily deduced when the trajectory *cDe* is not in one plane. In this case FIG. 204. let CE be the orthographic projection of the trajectory on the plane CTA, or *e*E be always perpendicular from the trajectory to CTA in E, and EK be the projection of the right line *ek* which represents the force that results from the action of the several centers upon the body at *e* with its direction. Then the second fluxions of EM and EN shall be measured by EL and EH;

EH; and the second fluxion of *eE*, the distance of the body from the plane CTA shall be measured by the difference of the perpendiculars *k*K and *e*E. From which it follows, that when a trajectory is described by a force that is directed towards a fixed point T, and another force that is always perpendicular to the plane CTA, the area described by TE about T on this plane flows uniformly.

FIG. 204. 470. Let a body set out from a given point D with a given velocity, and first let the trajectory be described in the plane CTA. Let the force at D that results from the attraction of the several centers be represented by D*ſ*, and be resolved into D*l* and D*b* parallel to TA and TC, respectively. Let DF and DG be perpendicular to TC and TA in F and G; upon which produced take F*f* equal to D*l*, and G*g* equal to D*b*. Let M*m* and N*n* be taken in the same manner upon EM and EN always equal to EH and EL, respectively. Let the velocity at D in the direction DG parallel to CT be such as would be acquired by a body falling from B to F with an uniform gravity equal to the force F*f*; and let the velocity at D in the direction parallel to TA be such as would be acquired in the same manner by a body falling from A to G with an uniform gravity equal to the force G*g*; or, in general, let these velocities be such as that half the square of each may be measured by the rectangles F*b* and G*a*, according to art. 434. Suppose likewise that while the body moves from D to E, TM decreases, but that EM increases, and that EK falls within the angle MEN. Then the square of the velocity of M (or of the fluxion of the base TM) shall be measured by 2BM*mfb*, and the square of the velocity of N (or of the fluxion of the ordinate EM) by 2G*a* — 2GN*ng*. For let M*q* measure the velocity of M, or the fluxion of CM, and the fluxion of this velocity (or second fluxion of CM) being measured by EH or M*m* (by art. 465.) it follows that the fluxion of the square of M*q* is equal to the fluxion of 2BM*mfb*, and that the square of M*q* is equal to 2BM*mfb*. In the same manner the square of the velocity of N is measured by 2G*a* — 2GN*ng*. And the fluxion of the base TM is to the fluxion of the ordinate EM in the subduplicate ratio of the area BM*mfb* to G*a*—GN*ng*. When the trajectory *c*D*e* is not in the plane CTA, the motions

at

at D and E are each to be reſolved into three motions in the directions parallel to CT, TA, and that which is perpendicular to CTA; the forces at D and E are each to be reſolved likewiſe into three forces in theſe directions; and the fluxions of TM, ME and E*e* are to be determined in the ſame manner as the fluxions of TM and ME in the former caſe.

471. There ariſes hence a ſimple enough conſtruction of the trajectory that would be deſcribed by the moon about the earth, if, the gravity towards the ſun being inverſely as the ſquare of the diſtance, the gravity towards the earth varied in the ſame proportion as the diſtance from its center, and we ſhould abſtract from the curvature of the earth's orbit during a revolution of the moon; which we ſhall ſubjoin, becauſe it ſerves to illuſtrate ſome parts of the theory of the moon's motion. For though the gravity towards the earth be not as the diſtance from its center, but inverſely as the ſquare of that diſtance; yet when the orbit of the moon is ſuppoſed to approach nearly to a circle, ſome of the effects of the ſolar action deduced from theſe ſuppoſitions will nearly coincide, and the trajectory is more eaſily determined in the former caſe than in the latter. Let S repreſent the ſun, T the earth, CADB the orbit of the moon about the earth, C and D the quadratures, A the conjunction, B the oppoſition, E the moon's place in this orbit at any time, and EN a perpendicular from E on ST that joins the centers of the ſun and earth. If the ſun acted on the earth and moon with equal forces, and in parallel lines, they would fall equally towards the ſun in parallel lines, and the ſolar action would not diſturb the motion of the moon and earth about their common center of gravity, or the motion of the moon about the earth, when the motion is all referred to the moon, as is uſual. But ſince the ſun acts with greater force upon the moon at A than upon the earth at T, and upon the earth at T with more force than upon the moon at B; it follows that, if the earth and moon fell towards the ſun, the moon would fall more than the earth towards the ſun in the former caſe, and the earth more than the moon in the latter; ſo that their diſtance from each other would increaſe by the inequality of the ſolar action in both caſes. If the moon fell either from C or D towards the ſun, and the earth from T, their Fig. 205.

diſtance

diſtance from each other would decreaſe by their falling in right lines that meet in the ſame center S. And it is obvious that the action of the ſun muſt diminiſh the force by which they tend towards each other where it would increaſe their diſtance if they fell towards the ſun, that is at A and B; but muſt increaſe that force where it would cauſe them to approach to each other, that is at C and D.

472. Let ST repreſent the gravity of the earth towards the ſun, and if S*f* be taken upon SE in the ſame proportion to ST as the ſquare of ST is to the ſquare of SE, S*f* will repreſent the gravity of the moon at E towards the ſun. Let *fg* parallel to ET meet ST in *g*, and the force S*f* being reſolved into S*g* (which acts at E in a right line parallel to TS) and *fg* which acts in the direction ET, Sir Isaac Newton neglects the part ST of the force S*g*, becauſe it is equal to the gravity of the earth towards the ſun, and acts in a parallel line; and, becauſe of the vaſt diſtance of the ſun, he conſiders *fg* as equal to ET, (the rather that as *fg* exceeds ET when E is in the part of the moon's orbit CAD, it is leſs than ET in the part DBC, and the exceſs in the former is nearly equal to the defect in the latter caſe when the angle CTE is the ſame;) and, S*f* being ſuppoſed to meet CD in *b*, *bf* is nearly double of E*b*, and T*g* nearly equal to 3TN. If TN be to N*n* as the ſquare of ST is to the ſquare of TN, the mean value of T*g* will be more nearly equal to 3TN + 10 N*n*; and a conſtruction of the trajectory may be derived from the preceeding articles upon this ſuppoſition likewiſe: but as this conſtruction is more complex, we ſhall ſuppoſe here (with Sir Isaac Newton) that T*g* is equal to 3TN; ſo that the trajectory may be ſuppoſed to be deſcribed by theſe three forces, the gravity of the moon towards the earth, a force directed towards the earth that is to the gravity of the earth towards the ſun as ET the diſtance of the earth and moon is to ST the diſtance of the earth and ſun, and a third force E*k* that acts in a right line parallel to TS, which is to the ſecond force as 3TN is to ET. Let G repreſent the gravity of the moon towards the earth at the diſtance TC, V the force *fg* which the ſolar action adds to this gravity when the moon is in quadrature to the ſun at C, S the periodic time of the earth about

bout the ſun, L the periodic time of the moon about the earth, *l* the time in which a circle would be deſcribed about T at the diſtance TC by G + V the whole force at C towards T, and I the time in which a circle would be deſcribed at the ſame diſtance by the force G only. Then by art. 432. G the gravity of the moon towards the earth is to the gravity of the earth towards the ſun in the ratio compounded of the ratio of TC to ST and of SS to 11: and the gravity of the earth towards the ſun is to V as ST is to TC; conſequently G is to V as SS is to 11. By the ſame article, 11 is to *ll* as G + V is to G; and therefore SS is to *ll* as G + V is to V. This being premiſed, the trajectory that would be deſcribed by the moon, if the gravity towards the earth, was varied in the ſame proportion as the diſtance and we ſhould abſtract from the angular motion of the right line TS, may be conſtructed in the following manner.

473. Firſt let the moon be ſuppoſed to ſet out from L in any direction LZ that is in the plane LTS; let the velocity of the motion be repreſented by LZ and be reſolved into LY parallel to CT and YZ parallel to TS; and let LF and LG be perpendicular to TC and TA in F and G. Upon the right line TC take TB the diſtance from T by falling from which to F a velocity would be acquired at F equal to LY, and TA the diſtance by falling from which to G in the right line AG a velocity would be acquired at G equal to YZ. From the center T deſcribe the circle BH*b* meeting FL in H, and the circle AI*a* meeting GL in I, and join TH and TI. Then in order to conſtruct the trajectory draw any right line TP meeting the circle BH*b* in P, draw TQ ſo that the angle ITQ may be always to the angle HTP in the ſubduplicate ratio of G — 2V to G + V, and let TQ meet the circle AI*a* in Q. Then PM a right line through P parallel to TA ſhall always interſect QN a right line through Q parallel to BT in E a point of the trajectory: And the ark LE of this trajectory ſhall be deſcribed in the ſame time that the ark HP is deſcribed by a body revolving with an uniform motion in the circle BP*b* by a force that is to G + V as TB is to TC. This conſtruction is deduced from art. 470. from which it follows that if Bb repreſent the whole force at FIG. 206.

B towards T and Tb meet PM in *m*, Aa the force at A towards T and Ta meet QN in *n*, the fluxion of the base TM shall be to the fluxion of the ordinate ME (or TN) in the subduplicate ratio of the trapezium BM*m*b to the trapezium AN*n*a, that is, in the ratio compounded of that of PM to QN and of the subduplicate ratio of the forces at equal distances from T in the right lines TC and TS or of $\sqrt{G+V}$ to $\sqrt{G-2V}$. The fluxion of the angle HTP is to the fluxion of ITQ in the ratio compounded of that of the fluxion of TM to the fluxion of TN and of that of QN to PM. Therefore the fluxion of the angle HTP is to the fluxion of ITQ as $\sqrt{G+V}$ to $\sqrt{G-2V}$; and these angles themselves are in the same ratio by art. 24. Whence the construction is manifest. We suppose V to be less than G in this construction. If V was equal to G, the trajectory would be the same that was constructed in art. 459. when the gravity was supposed to act in right lines perpendicular to a given right line and to be as the distance from it. If V was greater than G, the construction would depend on the logarithms, or on hyperbolic areas, but it would be of no use for our present purpose to describe that case.

474. The fluxion of TM is to the fluxion of HP, or the velocity of the point M to the velocity of P, as PM to TB; but the velocity of M is as PM; therefore the velocity of P is as TB, so that its motion is uniform and is the same with which the circle BHP is described by a force directed toward T that is to $G+V$ as TB is to TC. The motion of Q is likewise uniform and is the same with which the circle IQA is described when the centripetal force is directed towards T and is to $G-2V$ as TA to TC. And the arch LE is described in the trajectory in the same time that P with its uniform motion describes HP, or Q describes IQ. Let TK be taken upon TA in the same ratio to TN as 3V is to $G+V$. The gravity at E towards T is to G as ET to TC, by our supposition; the force added by the solar action at E is to V in the same ratio; and the sum of these forces is to $G+V$ in that ratio likewise. The third force E*k* (described in art. 472.) is to V as 3TN to TC, and to $G+V$ as TK to TC; therefore the force E*k* is to the sum of the other two forces as TK to ET; consequently the force that results from

from the three forces acts in the direction EK and is to G+V as EK to TC.

475. Let P*r* be perpendicular to TA in *r*, upon *r*P take *r*R to QN as $\sqrt{G-2V}$ is to $\sqrt{G+V}$, join TR, and let E*t* be perpendicular to TR from E; then E*t* shall be the tangent of the trajectory at E, and the velocity at E shall be to the constant velocity of P as TR is to TB. For the fluxion of TM is to the fluxion of TN as PM (or T*r*) to *r*R (by what was shewn in art. 473.) and to the fluxion of the curve LE as T*r* to TR; therefore the tangent intersects EN in an angle equal to RT*r*, (by prop. 14.) and is perpendicular to TR. The velocity at E is to the velocity of M as TR to T*r*; the velocity of M is to the velocity of P as T*r* to TP; consequently the velocity at E is to the constant velocity of P as TR to TB.

476. Let the body now set out from any point L in a right line L*s* that is not in the plane LTS, and its motion being represented by L*s* let it be resolved into a motion *s*Z in a direction perpendicular to the plane LTS and a motion LZ in that plane, and let the latter be resolved as above into two motions in the directions LY and YZ represented by these right lines. Then the rest of the construction being the same as in art. 473. let the motion *s*Z be such as a body would acquire by falling from D to T by the force that acts in the line of the quadratures BT. Let a circle D*d* described from the center T with the radius TD meet TP in X, from which let XV be perpendicular to TH in V; and the point E being determined as in art. 473. if E*e* be perpendicular to the plane LTS at E and equal to XV, *e* shall be a point in the trajectory. When the body sets out from any point that is not in the plane LTS; the points of the trajectory are determined by a similar construction. The point L is one of the *nodes* of the trajectory, or of the points wherein it intersects the plane LTS; and since XV vanishes when HP becomes equal to a semicircle or to any multiple of a semicircle, by the construction, it follows that the revolving body returns to the plane LTS every time the point P comes to the right line HT*b*; or that E becomes a node of the trajectory, and TE *the line of the nodes*, when HP becomes equal to a semicircle or to any multiple of a semicircle. Fig. 207.

477. If

477. If a rectangle *DG* be described so as to to touch the circle A*a* in the opposite points A and *a*, and the circle B*b* in the opposite points B and *b*, it appears from the construction that the curve described by the point E (which is the projection of the trajectory on the plane LTS) will touch the four sides of this rectangle in every revolution. The points in which it touches the sides G*g*, *D*d that are perpendicular to B*b* move backwards, and the points in which it touches the sides d*G* and g*D* that are perpendicular to A*a* move forwards, till they come to the angles *G* and *D*. The angles HTP, ITQ and PTQ increase uniformly; and when PTQ becomes equal to a right one if TP falls upon TA, the point E will pass through T, and the revolving body will pass above T at a distance that is to TD as the sine of the angle ATH is to the radius. Thus if H coincide
FIG. 208. with B, or the body set out from the plane LTS at the quadrature B, and the excess of $\sqrt{G+V}$ above $\sqrt{G-2V}$ be to $\sqrt{G+V}$ as unit is to any integer number *m*, the angle PTQ will be to BTP in the same ratio, PTQ will become a right angle after TP has described as many right ones as there are units in *m*, the point E will then pass through T, and the perpendicular E*e* will be equal to TD. The points in which the curve described by E touch G*g* and *D*d (which are in this case the *nodes* of the trajectory, or the points in which it intersects the plane LTS) in this time move with a retrograde motion from B to *G* and *b* to *D*. After this the motion of E becomes retrograde, the motion of the points where the curve LE touches G*g* and *D*d (or of the nodes of the trajectory) becomes direct; but the motion of the points where it touches d*G* and g*D* is then retrograde, and has always the same direction with the motion of E. And when TP has described as many right angles as there are units in 4*m*, the revolving body returns to B with its first velocity and direction. The inclination of the plane of the trajectory first increases, till it become perpendicular to the plane LTS when TP has described as many right angles as there are units in *m*; thereafter it decreases and returns to its first magnitude in an equal time. The revolutions in this trajectory from any quadrature to the subsequent quadrature on the same side of T are compleated in times equal to each other and to the

the time in which the point Q revolves in the circle A*a* by a force that is to G — 2V as TA is to TC. The revolutions from any conjunction to the ſubſequent conjunction are compleated in the ſame time that P revolves in the circle B*b* by a force that is to G + V as TB is to TC; and theſe times are to each other in the ſubduplicate ratio of G + V to G — 2V. What has been obſerved in this article holds only upon the ſuppoſition that the right line TS has no motion about the points T or S, and that the force towards T is as the diſtance; for when TS revolves about either of thoſe points, as in the caſe of the ſecondary planets, the nodes move backwards in every revolution, the inclination of the plane increaſes and decreaſes within narrower limits, and the other effects of the forces *fg* and T*g* are different.

478. It appears from the conſtruction likewiſe that when V is much leſs than G, H coincides with B, and TA is nearly equal to TB, the diſtance TE is leaſt not far from the octants before the ſyzygies, when TE biſects the angle BTA, and is greateſt after the other octants when TE biſects AT*b*. After ſome revolutions the figure of the trajectory differs little from an ellipſe that has its tranſverſe axis in the latter and the conjugate axis in the former poſition; for if the angle QTP was invariable, TA equal to TB, and QN parallel to BT interſected PM parallel to AT always in E (as in this conſtruction,) the curve deſcribed by E would be an ellipſe whoſe tranſverſe axis would biſect the angles AT*b* and *a*TB. The tides produced in the ocean by the inequality of the gravitation of the water towards the moon riſe to the greateſt heights about the ſame octants; of this ſee the latter part of *Corol.* 20. *Prop.* 66. *Lib.* I. *Princip.*

479. Suppoſe that the body ſets out from the node and quadrature B in a right line perpendicular to TB, and the reſt remaining as in art. 476. it is manifeſt from the conſtruction that the body will return to this node again, when the ray TP after compleating a revolution about T returns to the ſituation TB; in which time the ray TQ deſcribes an angle that is to four right ones in the ſubduplicate ratio of G — 2V to G + V. Therefore if the angle BT*q* be conſtituted on that ſide of TB which FIG. 208.

which is towards the opposition *a* in the ſame ratio to four right ones as the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G+V}$, and T*q* meet the circle A*a* in *q*, then *q*e be drawn parallel to TB and meet Be parallel to AT in e, the point e ſhall be the node of the trajectory after this revolution of P, and the angle BTe will ſhew how far the line of the nodes has moved backwards during this revolution. Let the motion at B be reſolved into ſZ perpendicular to the plane BTS (which repreſents the eclipτick) and Bz parallel to TA, and if the motion at B be given, or Bſ (which repreſents this motion) be invariable, the motion BZ will vary in the ſame ratio as the coſine of the angle ſBZ which is the inclination of the plane TBſ to the ecliptic. But the right line TA by falling along which this motion would be acquired varies in the ſame proportion as this motion, Be is equal to the ſine of the angle BT*q* the radius being T*q* or TA, and the angle BT*q* is given when the ratio of G to V is given; conſequently Be varies in the ſame ratio as the coſine of the inclination ſBZ; that is the tangent of the angle BTe is as the coſine of the inclination; and when V is much leſs than G, the angle BTe varies nearly in the ſame proportion. Therefore when the velocity at B with the ratio of G to V are given, the angle TBſ is right and V is much leſs than G, the retrograde motion of the node during this revolution is nearly as the coſine of the inclination of the plane.

480. In order to eſtimate from this conſtruction what ought to be the motion of the nodes of the moon nearly, we muſt ſuppoſe this trajectory to be as near to a circle as poſſible, and the body to revolve in it in the ſame time that the moon revolves about the earth. Firſt let the line of the nodes be in quadrature to the ſun about the middle of the month, and the inclination of the plane be ſuppoſed incomparably ſmall; then
FIG. 209 TA being ſuppoſed equal to TB (that the orbit may be nearly circular) and TB equal to TC, let the time in which the body E revolves from the quadrature in TB to the ſubſequent quadrature in the right line T*b* be equal to the time in which the moon revolves betwixt theſe quadratures. Then ſince XV vaniſhes when HP becomes equal to a ſemicircle, it follows that E revolves from the node in TB to the ſubſequent node *n* in the time

P makes

P makes half a revolution about T in the circle B*b*; but E revolves from the quadrature in TB to the subsequent quadrature in T*b* in the time Q makes half a revolution about T in the circle A*a*; and since the direct motion of the node while E proceeds from the right line T*n* to the quadrature in the right line T*b* is so exceeding small that it may be neglected (as will appear better afterwards,) it follows that the mean motion of the node is to the mean motion of the moon in this month as the difference of the times in which Q and P revolve about T is to the time in which Q revolves about T, that is as the difference of $\sqrt{G+V}$ and $\sqrt{G-2V}$ is to $\sqrt{G+V}$. Let S and L (as in art. 473.) represent the periodic times of the earth about the sun and of the moon about the earth; then since *l* is equal to the time in which P revolves about T in the circle B*b* by the force G + V, and L is equal to the time in which Q revolves about T, *ll* will be to LL as G — 2V to G + V. But by art. 473. SS is to *ll* as G + V is to V; consequently SS is to LL as G — 2V is to V; and G + V is to G — 2V as SS + 3LL is to SS. Therefore when the inclination of the orbit is incomparably small, and the orbit nearly circular, the mean motion of the node is to the mean motion of the moon in the month when the line of the nodes is in quadrature to the sun nearly as the excess of $\sqrt{SS+3LL}$ above S is to $\sqrt{SS+3LL}$. And if there are 2139 revolutions of the moon to the stars in 160 sidereal years (and consequently SS to LL in the duplicate ratio of these numbers) this proportion is that of 1 to 120,647. By the principles laid down in the excellent treatise concerning *the laws of the moon's motion according to gravity*, this proportion is about that of 1 to 120,639. If we would compare it with the ratio that results from Sir ISAAC NEWTON's method, their difference will appear very small if we may adapt his method to our present supposition in the following manner. It appears from what was shewn above (art. 432.) that when a force acts upon a body in a right line perpendicular to the direction of its motion, the deflexion of its course from a right line (or the angular velocity of the right line that is the direction of its motion and is always the tangent of the trajectory at the body) is as that force when the velocity is given, and in general

general as the force directly and velocity inversely. Hence (according to his method) the motion of the node when the moon is at the conjunction is to the inflexion of the course of the moon from a right line there, as the force that produces the motion of the node at the conjunction to the force that acts upon the moon there, that is as $3V$ to $G-2V$; and this inflexion of the orbit of the moon is to the inflexion of Q in describing the circle A*a* (which Q describes by the same force $G-2V$) as the velocity of Q is to the velocity of the moon at the conjunction, or to the velocity of P in the circle B*b*, that is as $\sqrt{G-2V}$ is to $\sqrt{G+V}$; so that, according to this method, the motion of the node at the conjunction A is to the motion of Q, or the mean motion of the moon, in the ratio compounded of the ratio of $3V$ to $G-2V$, and of the subduplicate ratio of $G-2V$ to $G+V$, that is in the ratio of $3LL$ to $S\sqrt{SS+3LL}$: According to which proportion the mean motion of the node in the month when the line of the nodes is in quadrature to the sun and the plane of the orbit is supposed almost coincident with the ecliptic, is to the mean motion of the moon as 1 to 120,643. And these three proportions agree nearly, the last of which is almost a mean betwixt the other two.

481. The mean motion of the node being determined when the planes are almost coincident, if this motion be diminished in the ratio of the cosine of the inclination of the plane of the moon's orbit to the radius, we shall obtain the mean motion of the nodes of the moon in this month nearly, by art. 479. If we diminish the motion of the node that was deduced at the end of the last article in the ratio of the cosine of $4^{\circ}.59'.35''$. (the inclination at the syzygies in this month) to the radius, the mean motion of the node in this month (according to this method) shall be to the mean motion of the moon as 1 to 121,1023, and the mean hourly motion of the node $16''\ 19'''\ \frac{3}{13}$. If we diminish the motion of the node deduced from our construction in the same ratio, the mean motion of the node will be to the mean motion of the moon as 1 to 121,10648, and the mean hourly motion of the node will be about $16''\ 19'''\ \frac{1}{7}$. The same hourly motion of the node by the principles that are pro-

proposed in the treatise of *the laws of the moon's motion* according to gravity is about 16″ 19‴ $\frac{3}{10}$. How much the motion of the node amounts to in any of the trajectories constructed in art. 476. in a revolution or any part of a revolution will appear from what follows.

482. Supposing first H to coincide with B as in art. 479. let P*l* perpendicular to TP meet TB in *l*, and *ln* parallel to AT meet the tangent at E (which is perpendicular to TR by art. 473.) in *n*, join T*n*, and it shall be the line of the nodes when the revolving body comes to *e* after describing the arch of the trajectory B*e*. For supposing the tangent of the trajectory at *e* to meet *t*E (the tangent of the curve described by E) in any point *n* and *nb* parallel to BT to meet PM in *b*; then because E*e* is always equal to XV, the fluxion of E*e* shall be to the fluxion of TV as TV is to XV; the fluxion of TV is to the fluxion of TM as XV is to PM; therefore the fluxion of E*e* is to the fluxion of TM as TV is to PM; consequently E*e* is to *bn* as TV to PM; and *bn* is to PM as E*e* (or XV) to TV, or as PM to TM, or M*l* to PM; so that *bn* is equal to M*l*, and *ln* parallel to TA; whence the construction is manifest. When P*l* becomes parallel to TB, the line of the nodes becomes parallel to the tangent at E, or perpendicular to TR. The velocity at *e* in the trajectory is to the constant velocity of P in the subduplicate ratio of the sum of the squares of TR and TV to the square of TB. The ratio of TA to TB is compounded of the ratio of the velocity in the trajectory at B to the velocity of P, the ratio of the cosine of the inclination of the plane to the radius, and that of $\sqrt{G+V}$ to $\sqrt{G-2V}$. FIG. 210.

483. Let *m*N be to QN as $\sqrt{G-2V}$ to $\sqrt{G+V}$, and *mo* parallel to TP meet P*l* perpendicular to TP in *o*, then the angle PT*o* shall be equal to BT*n* which shews how far the line of the nodes moves from the quadrature while the arch B*e* is described in the trajectory. For let *nb* meet AT in *u*, join *mu*, and let P*l* meet TA in *p*; then since the tangent E*n* is perpendicular to TR; PM is to R*r* (or *m*N) as *nb* (or M*l*) is to E*b*, or *u*N; consequently *m*N is to *u*N as R*r* to T*r*, and *um* is parallel to TP. Therefore *ln*, or T*u*, is to P*o* as T*p* is to P*p*, or as T*l* to TP, and the angle PT*o* is equal to *l*T*n*. Hence if the

arch bk be to a ſemicircle as the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G-2V}$, Tg be taken upon Tk in the ſame ratio to TA as $\sqrt{G-2V}$ to $\sqrt{G+V}$, and gf parallel to Tb meet bf perpendicular to Tb in f, the angle bTf ſhall be equal to the motion of the nodes in this trajectory in the time the body revolves from the quadrature B to the ſubſequent quadrature. When TA is equal to TB, and the planes are almoſt coincident; this angle bTf is to two right ones in the ratio compounded of the ratio of the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ to $\sqrt{G+V}$, and of that of the coſine of bTk to the radius nearly.

FIG. 211. 484. Suppoſe now that the line of the nodes is in any ſituation TL at the beginning of a revolution as in art. 476. and in order to determine the motion of the node during the revolution, take the arch Iq backwards from I in the ſame ratio to a whole circle as the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G+V}$, and let ql parallel to BT meet LF in l, join Tl, and the trajectory ſhall interſect the plane LTS in l after a compleat revolution of the point P. For when P returns to H, XV vaniſhes, and the arch deſcribed by Q is to a whole circle as $\sqrt{G-2V}$ to $\sqrt{G+V}$, by the conſtruction. Therefore when P returns to H, Q comes to q, and E to the point l in the plane LTS. The angle LTl ſhews how far the nodes have moved backwards in this time, and is to the angle ITq as the rectangle LGI to the ſquare of Tl nearly when V is much leſs than G. For let lx be perpendicular to Tl in x, and Io parallel to AT meet ql in o, then Lx ſhall be to Ll (or Io) as LG to TL, and Io to Iq as IG to TI nearly, conſequently Lx is to Iq as the rectangle LGI to the rectangle contained by TL and TI, and the angle LTl to ITq as the rectangle LGI to the ſquare of TL, nearly. Hence when TA is equal to TB, V is much leſs than G, and L is taken upon the circle Bb, if the ratio of V to G be given, the angle ITq is given, and the motion of the nodes (or the angle LTl) is as the ſquare of LG the ſine of the angle LTA the diſtance of the node from the ſun; which is agreeable to what is ſhewn of the nodes of the moon by Sir ISAAC NEWTON, *cor.* 2. *prop.* 30. *lib.* 3. *princip.* Suppoſing therefore the motion of the nodes of the moon to vary in the duplicate ratio of the ſine of their diſtance from the ſun, the annual motion of the nodes may be briefly

Fig. 207.

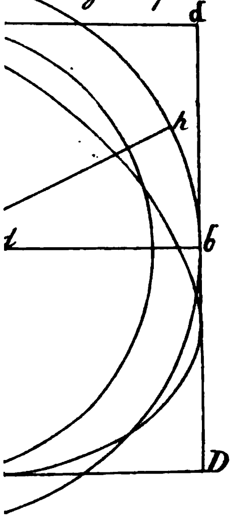
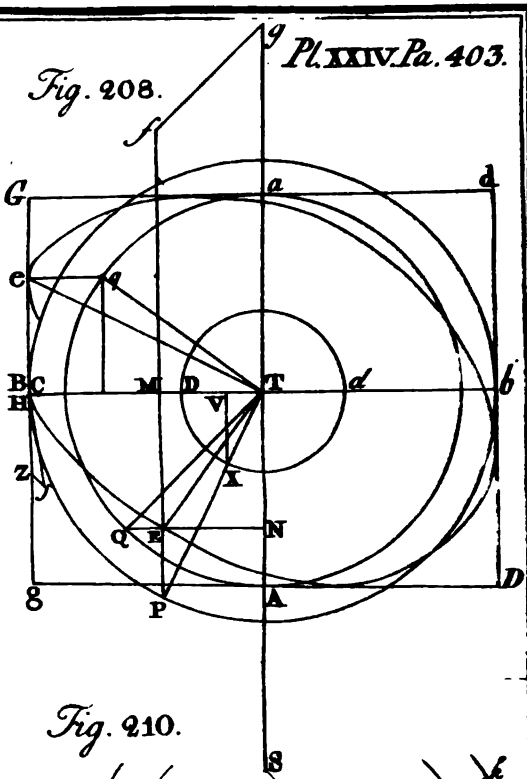


Fig. 208.



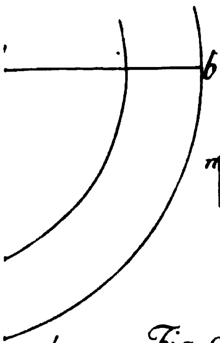


Fig. 210.

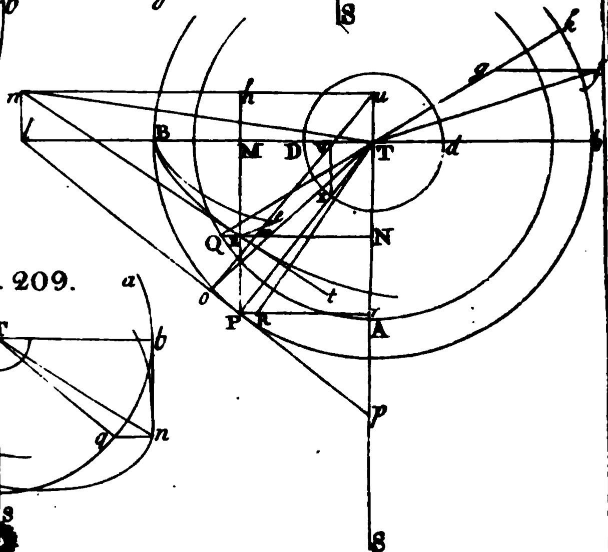
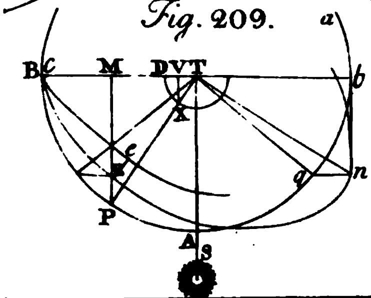


Fig. 209.



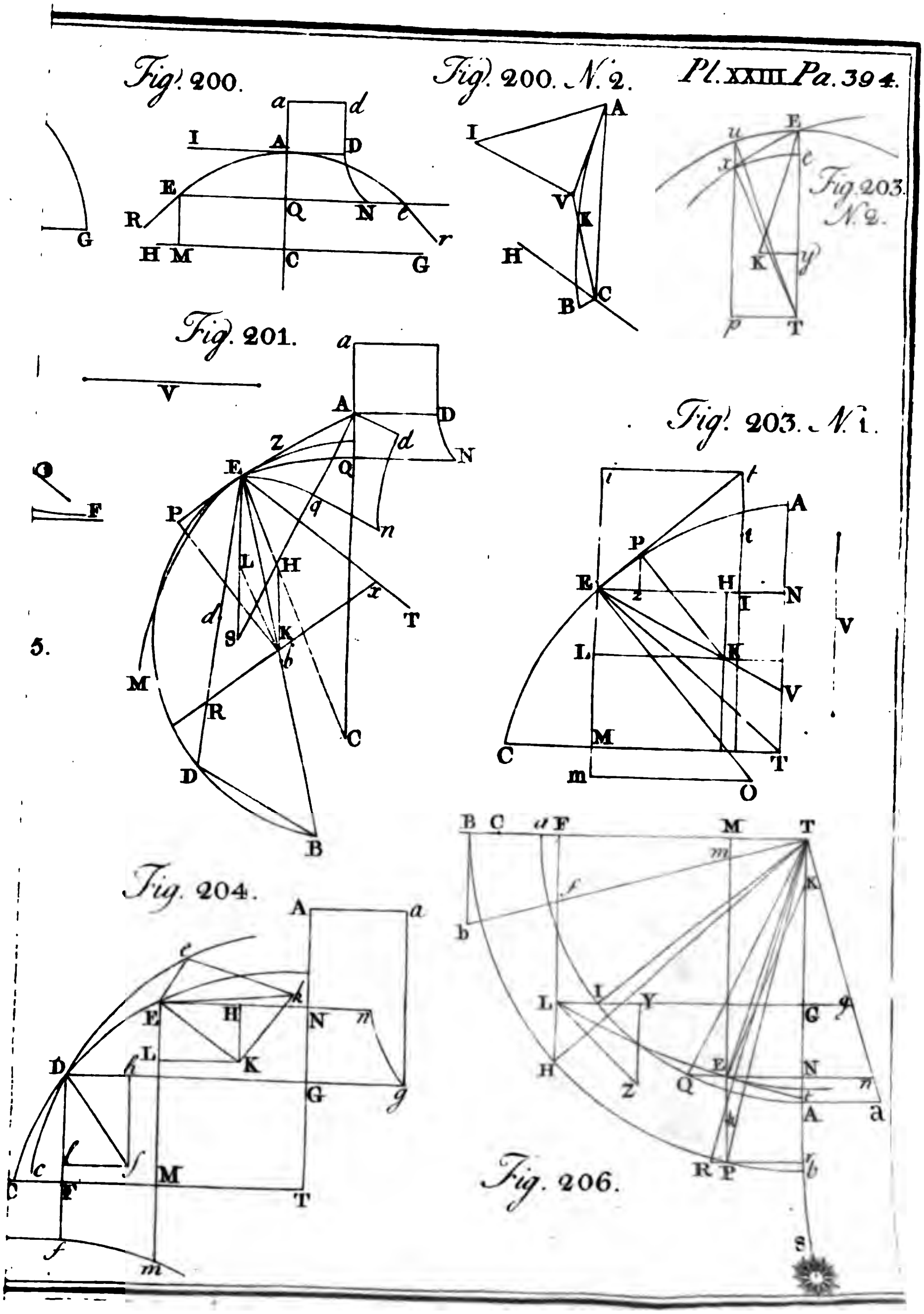
briefly computed from what was ſhewn in art. 481. by the uſeful theorem in the ſcholium of *prop.* 33. *lib.* 3. *princip.* communicated by Mr. MACHIN and ſince explained in a more general manner in the *laws of the moon's motion*, &c. *p.* 14. By that theorem, the mean motion of the ſun from the node is a geometrical mean proportional between the motion of the ſun and the mean motion of the ſun from the node in the month when the line of the nodes is in quadrature to the ſun The mean motion of the node in this month is to the motion of the ſun, according to the firſt proportion deduced in art. 481. in the compound proportion of 1 to 121,1023, and of the mean motion of the moon to the mean motion of the ſun, or of 2139 to 160, that is in the proportion of 1 to 9,05861; conſequently the mean annual motion of the node is to the motion of the ſun (by the theorem) as 1 to 18,60413; which gives 19° 21′ 2″ for the motion of the node in a ſydereal year. According to the ſecond proportion in art. 481. that was deduced from our conſtruction, this motion is about 19° 21′; by the principles that are laid down in the above mentioned treatiſe it is 19° 21′ 7″ ½; and by aſtronomical obſervation it is 19° 21′ 21″ ½. In theſe computations we have abſtracted from the acceleration of this motion that ariſes from the exceſs of the ſolar force on that half of the moon's orbit which is towards the ſun above what it is on the other half, and from any effect the excentricity of the lunar orbit, and the increaſe of the inclination of the plane while the nodes move from the quadratures to the ſyzygies, may be ſuppoſed to have upon it. But this is ſufficient for our purpoſe, which is only to illuſtrate from the conſtruction of the trajectory (in art. 476.) the computation of this motion from its cauſe, which affords ſo remarkable a confirmation of the principle of gravity, and may on other accounts be of great uſe. See the *laws of the moon's motion*, *p.* 24.

485. In general let the body ſet out from the node L as in art. 476. and let it be required to determine the ſituation of the line of the nodes when it comes to any point *e* in the trajectory. The tangent E*t* of the curve LE is perpendicular to TR by art. 475. Let the angle TP*p* be made equal to the angle ATH and P*p* meet TB in *p*; let *pn* parallel to TA FIG. 212.

meet E*t* in *n*, join T*n*, and the nodes ſhall be in the right line T*n* when the body comes to *t* after deſcribing the arch L*e* in the trajectory. The demonſtration is ſimilar to that in art. 482. If *m*N be taken upon NQ in the ſame ratio to NQ as $\sqrt{G-2V}$ is to $\sqrt{G+V}$, and *mo* be perpendicular to P*p* in *o*, the angle HT*n* will be equal to *o*TP, and LT*n* the motion of the node will be either equal to the ſum or difference of the angles *o*TP and LTH. When TP becomes perpendicular to TH, P*p* becomes parallel to BT and the line of the nodes perpendicular to TR.

486. In art. 479. we ſuppoſed the body to ſet out from B in a right line perpendicular to TB. But if it ſet out from any FIG. 213. point L in TB, and the right lines TB, TL with the velocity at L be given, the motion of the node will be ſtill as the ſine-compliment of the inclination of the plane. If TA be equal to TB, and the arch BH be taken from B towards the oppoſition *a* in the ſame ratio to a ſemicircle as the exceſs of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G+V}$, the line of the nodes will deſcribe an angle equal to BTH in the time P deſcribes the ſemicircle H*b*; becauſe P and Q will come at the ſame time to the point *b*. This is the motion of the node that was determined in the firſt part of art. 180. Suppoſing ſtill the node L to be in the line of the quadratures TB, let the angle BTH be to a right angle as the difference of $\sqrt{G+V}$ and $\sqrt{G-2V}$ is to $\sqrt{G-2V}$, (or in the caſe of the moon according to art. 180. as 1 to 119,6469) and the arch BH be taken from B towards the oppoſition; let the point P deſcribe the circle B*b*, and Q the circle LA, TA being ſuppoſed equal to TL; then (by art. 476.) the right lines TP and TQ will coincide with each other at the conjunction, where the motion in the trajectory will become perpendicular to TA; and (by art. 485.) while the body moves from the quadrature L to the conjunction A, the line of the nodes will deſcribe an angle equal to BTH. But if we ſuppoſe a body to ſet out from the ſame point L in a right line perpendicular to TL, (as in art. 479.) the point P to deſcribe the circle LA and Q the circle B*b*, ſo that the diſtance of the body from T at the conjunction may be equal to TL its diſtance at the quadrature, the motion of the node while the body moves from the

Pl. XXIII Pa. 394.



the quadrature to the conjunction will be equal to the angle ATR if the angle BTA be to BTq as $\sqrt{G+V}$ to $\sqrt{G-2V}$, and qn be to Rn in the same ratio, by art. 482. and this angle ATR is to a right one as 1 to 121,65526. If we suppose the motion of the node in a circular orbit, when the planes are almost coincident, to be a mean betwixt the motion of the node in those two cases, it will be to the mean motion in the orbit as 1 to 120,64273. This almost coincides with the proportion that was deduced for this case at the latter end of art. 180. in imitation of Sir ISAAC NEWTON's method; which in larger numbers is that of 1 to 120,64275.

487. The angle in which the plane of the lunar orbit intersects the ecliptick is perpetually varying; and this motion may be also illustrated from art. 476. In general if EK be perpendicular to the line of the nodes Tn in K, the inclination of the plane of the trajectory to the plane BTA is an angle the tangent of which is to the radius as XV to EK. First let the line of the nodes be in quadrature to the sun, and suppose the body to set out from the quadrature in a right line perpendicular to TB, as in art. 479. make the angle STb equal to the inclination at the quadrature; take TZ upon Tb in the same ratio to TB as the velocity at B in the trajectory is to the velocity of P in the circle Bb; let Zd parallel to ST and Td perpendicular to ST meet in d; upon dZ take da to dZ as $\sqrt{G+V}$ is to $\sqrt{G-2V}$, join Ta, and the angle STa will be nearly equal to the inclination at the syzygies. For TZ is equal to the semi-axis that is conjugate to TB in the ellipse that would have been described about T if the third force Tg (described in art. 472.) had not acted; and TA is to dZ as $\sqrt{G+V}$ to $\sqrt{G-2V}$: consequently TA is equal to da. But at the syzygies the distance of the point E from T is nearly equal to TA, and Ee to Td. Therefore the angle STa is the inclination at the syzygies nearly, and the angle aTZ is the decrement of the inclination while the body moves from the quadratures to the syzygies. Let Zy be perpendicular to Ta in y, and because Zy is to Za as Td is to Ta, the angle aTZ is nearly to Tad (or STa) as Za is to TA, that is in the ratio compounded of that of za to da (or of the excess of $\sqrt{G+V}$ above $\sqrt{G-2V}$ to $\sqrt{G+V}$) and that of da to Ta:

FIG. 212.

FIG. 214. *and* 209.

Ta (or of the cosine of the inclination to the radius.) But the mean motion of the node in this revolution is to the mean motion of E in the same ratio, by art. 481. Therefore (by applying this to the moon) the excess of the inclination at the quadratures above what it is at the syzygies in this month is to the latter inclination as the mean motion of the node in this month is to the mean motion of the moon, or as 1 to 121,1; which is the ratio assigned in the treatise above mentioned, *p.* 26. where the intermediate inclinations are also determined in an elegant manner. According to this proportion the difference of the inclinations at the quadratures and syzygies is about $2' 28'' \frac{3}{7}$; and if this difference be increased in the ratio of the synodical to the periodical month, it will then agree nearly with that which results from Sir Isaac Newton's method, *prop.* 33. *lib.* 3. *princip.* If the body be supposed to continue its motion in the trajectory, the inclination at the node e will be greater than the inclination at the node B at the beginning of the revolution in the same ratio as the radius is greater than the sine of an arch that is to a circle as $\sqrt{G-2V}$ to $\sqrt{G+V}$.

Fig. 215. 488. Suppose O to be the projection on the plane BTA of *o* the place where the distance of the body that describes the trajectory from the plane BTA is greatest at the beginning of any revolution; and let the arch OQ be taken backwards from O on the circle OK described from the center T in the same ratio to the whole circumference as the excess of $\sqrt{G+V}$ above $\sqrt{G-2V}$ is to $\sqrt{G+V}$; let QE parallel to BT meet OE parallel to TA in E, and TE meet the arch OQ in V; then if TD perpendicular to TV be equal to O*o*, the angle EDV will shew nearly how much the inclination of the plane varies in this revolution; because the angle TVD is equal to OT*o*, and TED is nearly equal to the inclination of the plane when the body after almost a compleat revolution is again at the greatest distance from BTA. When TO is within the angle BTA, or *b*T*a*, TE is less than TO, the angle TVD is less than TED, and consequently the inclination of the plane increases: but when TO is within the angle AT*b* or *a*TB, TE is greater than TO, and the inclination of the plane decreases; that is while the line of the nodes (which is perpendicular to TO) moves from the quadratures

dratures to the ſyzygies, the inclination of the plane increaſes; but while the line of the nodes moves from the ſyzygies to the quadratures, the inclination decreaſes.

489. Let the arch E*x* deſcribed from the center D meet DV in *x*, and VG be perpendicular to TA in G; then becauſe E*x* is to EV as TD to DV, EV to VQ as VG to TG, and VQ to OQ as the ſquare of TG to the ſquare of TV, nearly, it follows that the angle EDV is to OTQ, in the ratio compounded of that of TD to DV, and that of the rectangle TGV to the rectangle contained by TV and DV. But the angle OTQ is to the motion of the node in that month when the line of the nodes is in quadrature to the ſun (which we ſhall call N) as DV to TV, by art. 479. conſequently the angle EDV is to N in the ratio compounded of the ratio of TD to DV, and that of the rectangle TGV to the ſquare of TV, that is in the ratio of the rectangle contained by the ſine of the inclination and the ſine of 2BTV (the double diſtance of the nodes from the ſun) to twice the ſquare of the radius; which agrees with *cor.* 3. *prop.* 33. *lib.* 3. *princip.* The decrement of the inclination in a revolution while the nodes move from the ſyzygies to the quadratures is nearly as EV; it is greateſt when V is in the octants, where EV becomes nearly equal to $\frac{1}{2}$ OQ; and EDV, the decrement in a revolution, is to the mean motion of the nodes nearly as TD to DV, or as the ſine of inclination to the radius. Becauſe the radius TO and the arch OQ are ſuppoſed to be given, the point E is always in an ellipſe HE*b*, whoſe tranſverſe axis is in the octants after the ſyzygies; and the whole decrement of the inclination while the nodes move from the ſyzygies to the quadratures is nearly to the decrement that would have been generated in the ſame time by the variation at the octants continued uniformly, as the area deſcribed by EV while O moves along the quadrant K*k* is to the rectangle contained by $\frac{1}{2}$ OQ and K*k*, or as the area included betwixt the quadrants of the ellipſe and circle to that rectangle, that is as the radius TK to the quadrant K*k*. Hence the mean hourly decrement of the inclination is to the mean hourly motion of the node as the ſine of the inclination (the radius being TK) is to the quadrant K*k*; ſo that when the inclination is ſmall, the whole

whole inclination would be generated by its mean hourly variation in the ſame time that a quadrant would be deſcribed by the nodes with their mean motion. Therefore the whole decrement of the inclination while the nodes move from the ſyzygies to the quadratures (or while a quadrant is generated by the motion of the ſun from the node) is to the whole inclination as the mean motion of the node is to the mean motion of the ſun from the node, or as 1 to 19,6 (by art. 484.) according to which proportion that decrement is about 16′ 10″. See the *laws of the moon's motion*, &c. *p.* 26.

FIG. 216. 490. If we ſuppoſe a body to deſcend along the quadrant BA, and the velocity at B to be equal to the velocity of Q as in art. 480. the velocity at A will be equal to that which would be acquired by falling in the chord from B to A or by falling from B to T and then from T to A, by art. 362. The velocity that would be thus acquired at A is the ſame that would be acquired if the motion was not accelerated from B to T, and was accelerated from T to A by the force T*g* (*Fig.* 204) only. Let B*b* perpendicular to TB be to A*a* perpendicular to TA as $\sqrt{G-2V}$ is to $3V$, and the velocity acquired at A will be to the velocity at B in the ſubduplicate ratio of the ſum of the triangles TB*b*, TA*a* to the triangle TB*b*, or of the ſum of B*b* and A*a* to B*b*, that is as $\sqrt{G+V}$ to $\sqrt{G-2V}$ or as $\sqrt{SS+3LL}$ to S. In like manner the velocity at any point E is determined. And this will be found to agree nearly with what is ſhewn concerning the acceleration of the area deſcribed about the earth in a circular orbit, *prop.* 26. *lib.* 3. *princip.* where this acceleration is computed by the increment of the velocity in ſuch an orbit; and it coincides with what is ſhewn *p.* 29 of the *laws of the moon's motion.* The ſame increaſe of the velocity may be deduced from what has been ſhewn of the trajectory B*e* when its
FIG. 209. plane coincides with the plane BTS, and it is ſuppoſed nearly circular; for the velocity at B is equal to the velocity of Q; when TA is equal to TB the velocity at the ſyzygies is nearly equal to the velocity of P, by art. 475. and theſe velocities are to each other as $\sqrt{G-2V}$ to $\sqrt{G+V}$. Some other corollaries relating to this theory might be deduced from the preceeding articles.

491. If

491. If a fluid be ſuppoſed to gravitate towards two points C and S with equal forces that are the ſame at all diſtances, the figure of the fluid will be an oblong ſpheroid, and theſe two points will be the *foci* of the generating ellipſe. For ſuppoſing AE*a* to be any ſection of the fluid through C and S, let EM and EN be taken upon EC and ES repreſenting the equal forces towards the points C and S reſpectively; let MP and NQ be perpendicular to the tangent at E in P and Q, and EP will be equal to EQ becauſe of the equilibration of the fluid: The fluxion of CE is to the fluxion of the curve AE as EP to EM, and the fluxion of SE to the fluxion of AE as EQ to EN, by prop. 17. Therefore CE and SE flow with equal motions, and (becauſe SE decreaſes while CE increaſes) the ſum of CE and SE is invariable; conſequently AE*a* is an ellipſe that has its *foci* in C and S. The gravity at E is to the gravity at A as the ſine of the angle CEP is to the radius. When the forces directed towards C and S are invariable but unequal, let D be any point of the ſurface AE*a* that terminates the figure, and it will appear in the ſame manner that the difference of CE and CD will be to the difference of SD and SE as the force towards C is to the force towards S. The figures by which rays of light iſſuing from a given point point S may be refracted ſo as to have their *focus* afterwards in C are of this kind. The figure AE*a* is in ſome caſes a conic ſection, and when it paſſes through the point C or S it is a portion of an epicycloid that is deſcribed when a circle revolves on an equal circle. If equal and invariable forces are directed towards any number of given centers, the aggregate of the right lines drawn from any point in the ſurface that terminates the fluid to thoſe centers is invariable. When the forces towards the centers are inverſely as the ſquares of the diſtances, the aggregate of right lines that are inverſely as the diſtances of any point of the ſurface from thoſe centers is invariable. And the figures have an analogous property when theſe forces at equal diſtances are not equal but in a given ratio. But it would be of little uſe in philoſophy to inſiſt on this ſubject. Fig. 217.

492. Suppoſe that a fluid, which gravitates towards the point C, revolves about the axis AB; let ER be perpendicular Fig. 218.

to AB in R from any point E in AEB the furface of the fluid, and let CD be the ordinate at C; then it will appear (as in the laft article) that the fluxion of CE fhall be to the fluxion of ER as the centrifugal force is to the gravity at E, or (becaufe the centrifugal force at E is to the centrifugal force at D as ER is to CD) in the compound ratio of the centrifugal force at D to the gravity at E and of ER to CD. From which it follows that if the gravity towards C be the fame at all diftances, and a right line L be to CD as the gravity is to the centrifugal force at D, the fquare of ER will be equal to the rectangle contained by CE — CA and $\frac{1}{2}$ L, which agrees with what has been fhewn by Mr. HUYGENS. If the gravity be inverfely as the fquare of the diftance from C, let G reprefent the gravity and V the centrifugal force at D, and let the fquare of the right line L be to the fquare of CD as G + $\frac{1}{2}$ V is to $\frac{1}{2}$ V, then the folid contained by CE and the fquare of ER fhall be equal to the folid contained by CE — CA and the fquare of L, and CD will be to CA as G + $\frac{1}{2}$ V is to G. But the figure of the earth or planets is not to be difcovered by fuppofitions of this kind; for as the gravity of a body refults from the gravity of all its parts, fo the gravitation towards any body refults from the gravitation towards the particles of which it confifts, as is fhewn by Sir ISAAC NEWTON; and when the figure of the fluid varies from a fphere, this gravitation is not directed towards a fixed point. Of this we fhall treat in the next chapter, where we fhall fhew that if the earth was of an uniform denfity, its figure would be an oblate fpheroid accurately.

493. We have infifted at fo great length on what relates to the curvature of lines, and to the application of this theory to philofophical problems, becaufe in this confifts one of the greateft advantages of the modern geometry. There are many other problems that depend on this theory, but we fhall conclude this fubject with obferving, that as when a right line interfects an arch of a curve in two points, if by varying the pofition of that line the two interfections unite in one point, it then becomes the tangent of the arch; fo when a circle touches a curve in one point and interfects it in another, if by varying the center this interfection join the point of contact, the circle then

then has the closest contact with the arch and becomes the circle of curvature; but it still continues to intersect the curve at the same point where it touches it (that is where the same right line is their common tangent, according to the definition in art. 181.) unless another intersection join that point at the same time. Let ME*m* be any arch that is continued from E on both sides of the ray of curvature CE, ACB a right line perpendicular to CE; let the right line CM revolve about C and meet the curve in M, and PM be perpendicular to AB in P: then if AP be supposed to flow uniformly, the first and second fluxions of CM vanish when M comes to E; and, when the third fluxion of CM, or of PM, does not vanish at the same time, the circle of curvature intersects the curve at E. In general when M comes to E, and the number of the fluxions of CM of successive orders (including the first fluxion) which then vanish is even, the circle of curvature at E intersects the arch ME*m*; but if this be an odd number, there is no such intersection at E. For let PN be taken upon PM always equal to CM. Then the curve NE*n* described by N will pass through E; and it is manifest, that the circle of curvature intersects the arch ME*m* only when NE*n* intersects its tangent at E, that is when NE*n* has a point of contrary flexure at E. Therefore (by art 266.) the circle of curvature at E intersects the arch ME*m* only when the number of those fluxions of CM that vanish at the term of the time when M comes to E, is an even number. This theorem may serve to illustrate a subject that was disputed some time ago by two celebrated authors. One of them imagined that two points of contact, or four intersections, of the curve and circle of curvature must join each other to form an osculation. But Mr. James Bernovilli insisted, on just grounds, that the coalition of one point of contact and one intersection, or of three intersections, was sufficient. In which case (and in general when an odd number of intersections only join each other) the point where they coincide continues to be an intersection of the curve and circle of curvature, as well as a point of their mutual contact and osculation. Fig. 219.

494. In the preceeding chapters we deduced the principal propositions of the method of fluxions, and those upon which

its application to the resolution of problems depends, from the plain axioms concerning motion in art. 15. Having proposed this doctrine (art. 2.) as a method of deducing the relations of quantities, by comparing the motions which are conceived to generate them, it seem'd natural to establish this method on the most evident principles that relate to motion; and if it be no less known to us than extension, it would seem, that as this doctrine is far more general and comprehensive than the common geometry, so it cannot be said to be inferior to it in accuracy or evidence. Nor can it seem improper, to deduce the properties of figures from the same principles which serve for describing their genesis. Thus, the definition of a fluxion (art. 11.) seem'd naturally to lead us into the method of treating this doctrine which we have followed hitherto; and its connexion with the manner in which the genesis of figures is most commonly described in geometry, and with the most useful theories concerning motion in philosophy, (of which we have had some examples in this chapter,) induced us the rather to pursue it. But we have insisted on it at so great length chiefly, because a full account of the manner in which the principal propositions of the method of fluxions are demonstrated by it, may be of use for removing several objections that have been lately urged against this doctrine; which has been represented, as depending on nice and intricate notions; while it has been insinuated, that they who have treated of it have been earnest rather to go on fast and far, than solicitous to set out warily, and see their way distinctly. But we now proceed to the more concise methods by which the fluxions of quantities are usually determined.

The End of the first Volume.

Fig. 212

Fig. 213.

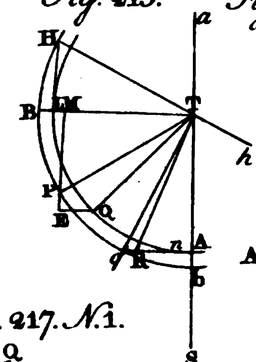


Fig. 214.

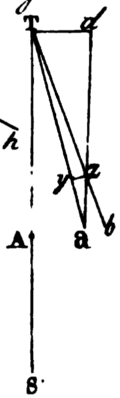


Fig. 215.

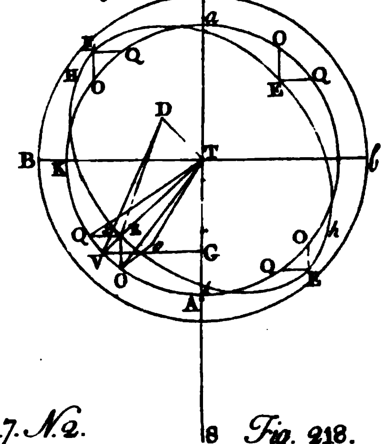


Fig. 217. N. 1.

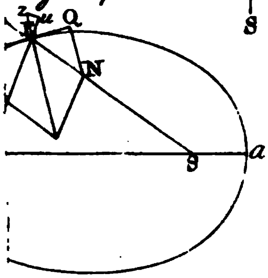


Fig. 217. N. 2.

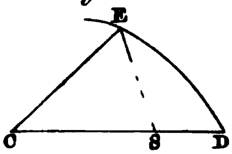


Fig. 218.

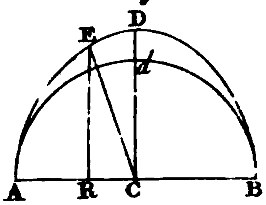


Fig. 220.

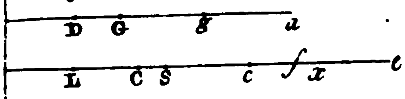


Fig. 225.

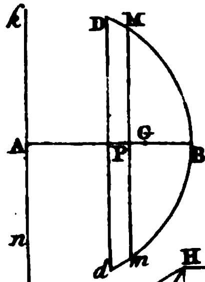


Fig. 222.

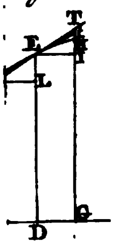


Fig. 223.

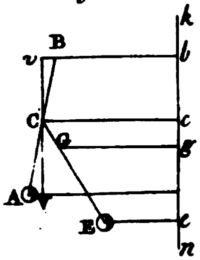


Fig. 224.